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Multisummability of Formal Solutions of Some Linear Partial Differential Equations

Sunao Ōuchi

Department of Mathematics, Sophia University, Tokyo 102-8554, Japan
E-mail: ouchi@mm.sophia.ac.jp

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Let $\hat{u}(t,x)=\sum_{n=1}^{\infty} u_n(x)t^n$ be a formal power series in one variable t satisfying formally $P(t,x,\partial_t,\partial_x)u(t,x)=f(t,x),(t,x)\in\mathbb{C}\times\mathbb{C}^d$. The purpose of the present paper is to introduce a class of linear partial differential operators and show that $\hat{u}(t,x)$ is multisummable provided $P(t,x,\partial_t,\partial_x)$ belongs to this class.

Key Words: multisummability; formal solutions, asymptotic expansion.

INTRODUCTION

The theory of the multisummability of formal power series has recently developed. It is applied to ordinary equations and gives fruitful results, in particular, it is shown that solutions of formal power series of ordinary differential equations are multisummable (see [2, 4, 5]). In the present paper we treat solutions of formal power series of a linear partial differential equation in $\mathbb{C}^{d+1}$,

$$P(t,x,\partial_t,\partial_x)u(t,x)=f(t,x), \qquad (t,x)\in\mathbb{C}\times\mathbb{C}^d. \tag{0.1}$$

We will often construct a solution of formal power series in one variable t of (0.1), $\hat{u}(t,x)=\sum_{n=0}^{\infty} u_n(x)t^n$, where $u_n(x)$ is holomorphic in a neighborhood of $x=0$ in $\mathbb{C}^d$. The relation between formal solutions and genuine solutions with asymptotic expansions is an important problem. It is shown in [9, 10] that solutions with some growth estimates of partial differential equations in some class have asymptotic expansions. The existence of genuine solution with asymptotic expansion $\hat{u}(t,x)$ is studied in [7]. However, the multisummability of formal solutions is not investigated in these papers. As for multisummability of solutions of formal power series of partial differential equations, the situation is quite different and we know a little results. The multisummability of solutions of formal power series are firstly studied in

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[6], where formal solutions of the initial value problem of the heat equation are considered and the conditions of their multisummability are given by those of initial values. Their results suggest that, in general, solutions of formal power series of partial differential equations are not multisummable unless strict conditions are satisfied.

The aim of this paper is to find a class of linear partial differential operators and to show the multisummability of the formal solution $\hat{u}(t,x)$ of (0.1) for $P(t,x,\partial_t,\partial_x)$ belonging to this class. The contents of the present paper are as follows. In Section 1 we sum up Laplace transform, Borel transform and multisummability of functions. In Section 2 we define the characteristic polygon (Newton polygon) of $P(t,x,\partial_t,\partial_x)$ with respect to $t=0$ and introduce a class of operators by using the characteristic polygon. Roughly speaking, this class consists of partial differential operators which are considered as perturbations of ordinary differential operators in some sense. It is the main result that the multisummability of formal solutions holds for operators in this class. We give the proof of main result (Theorem 2.3) in the following sections. We use convolution equations to show it, as was done in [4, 5]. In Section 3 we study formal partial differential equations, so the discussions are formal. In Section 4 we introduce convolution equations derived from the original equation and study the existence and growth estimate of solutions, from which Theorem 2.3 follows. However, we leave Lemma 4.3 and Proposition 4.4 not proved, so we give the proofs of them in Sections 5 and 6.

1. LAPLACE TRANSFORM, BOREL TRANSFORM AND MULTISUMMABILITY

In this section we give notations and definitions. The set of all nonnegative integers is denoted by $\mathbb{N}=\{0,1,2,\dots\}$. The coordinates of $\mathbb{C}\times\mathbb{C}^d$ are denoted by $(t,x)=(t,x_1,x_2,\dots,x_d)$ and $|x|=\max\{|x_i|;1\leqslant i\leqslant d\}$. The differentiations are denoted by ∂_t, ∂_{x_i} and $\partial_x=(\partial_{x_1},\partial_{x_2},\dots,\partial_{x_d})$. For a multiindex $\alpha=(\alpha_0,\alpha')=(\alpha_0,\alpha_1,\dots,\alpha_d)\in\mathbb{N}\times\mathbb{N}^d$, $|\alpha|=\alpha_0+|\alpha'|=\sum_{i=0}^d\alpha_i$ and $\partial_x^{\alpha'}=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}$. For open sets V and U, $V\Subset U$ means that $\bar{V}$ is compact and $\bar{V}\subset U$.

For $\theta\in\mathbb{R}$ and $\delta>0$ $S(\theta,\delta)=\{t\neq0;|\arg t-\theta|<\delta\}$ is a sector in t-space and $S^*(\theta,\delta)=\{\xi\neq0;|\arg\xi-\theta|<\delta\}$ is a sector in ξ-space. For $S(\theta,\delta)$ ($S^*(\theta,\delta)$), $S_{\{0\}}(\theta,\delta)=\{t\in S(\theta,\delta);0<|t|<\rho(\arg t)\}$ (resp. $S^*_{\{0\}}(\theta,\delta)=\{\xi\in S^*(\theta,\delta);0<|\xi|<\rho(\arg\xi)\}$), where $\rho(\cdot)>0$ is some positive continuous function on $(\theta-\delta,\theta+\delta)$, which is called a neighborhood of $t=0$ in $S(\theta,\delta)$ (resp. $\xi=0$ in $S^*(\theta,\delta)$). For an open set Ω, $\mathcal{O}(\Omega)$ is the set of all holomorphic functions on Ω. $\mathcal{O}(\Omega)[[t]]$ is the set of all formal power series in one variable t with coefficients in $\mathcal{O}(\Omega)$.

Now let us define Laplace transform, Borel transform and multi-summability of functions. We refer the details of these topics and the proofs of some lemmas to Balser [1]. Let U be a neighborhood of the origin in $\mathbb{C}^d$ and $\gamma > 0$ be a constant. Given θ and $\delta > 0$, set $S^* := S^*(\theta, \delta)$ and $S^*_{\{0\}} := S^*_{\{0\}}(\theta, \delta)$.

DEFINITION 1.1. $\mathrm{Exp}(\gamma, S^* \times U)$ is the set of all $\phi(\xi, x) \in \mathcal{O}(S^* \times U)$ such that

$$|\phi(\xi, x)| \leqslant A \exp(c|\xi|^\gamma) \quad \text{for } (\xi, x) \in S^* \times U \text{ with } |\xi| \geqslant 1 \tag{1.1}$$

holds for some positive constants A and c.

Let $\phi(\xi, x) \in \mathrm{Exp}(\gamma, S^* \times U)$ satisfying

$$|\phi(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma} (\varepsilon > 0) \quad \text{in } \{(\xi, x) \in S^* \times U;\ 0 < |\xi| < \rho_0\}. \tag{1.2}$$

Then we can define the γ-Laplace transform $(\mathcal{L}_{\gamma,\theta}\phi)(t, x)$ by

$$(\mathcal{L}_{\gamma,\theta}\phi)(t, x) = \int_0^{\infty e^{i\theta}} \left(\exp - \left(\frac{\xi}{t}\right)^\gamma\right) \phi(\xi, x)\, d\xi^\gamma. \tag{1.3}$$

$(\mathcal{L}_{\gamma,\theta}\phi)(t, x)$ is holomorphic in $S_{\{0\}}(\theta, \pi/2\gamma + \delta) \times U$. Let $\psi(t, x)$ be holomorphic in $S_{\{0\}}(\theta, \pi/2\gamma + \delta) \times U$ and $|\psi(t, x)| \leqslant C|t|^c$ for some $c \in \mathbb{R}$. Let $\xi \neq 0$ with $|\arg \xi - \theta| < \delta$ and $\mathscr{C}$ be a contour in $S_{\{0\}}(\theta, \pi/2\gamma + \delta)$ from $0 \exp(i(\theta' + \arg \xi))$ to $0 \exp(i(-\theta' + \arg \xi))$ with $\pi/2\gamma < \theta' < \pi/2\gamma + \min\{\theta + \delta - \arg \xi,\ \arg \xi - \theta + \delta\}$. Then we define the γ-Borel transform $(\mathcal{B}_{\gamma,\theta}\psi)(\xi, x)$ by

$$(\mathcal{B}_{\gamma,\theta}\psi)(\xi, x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \left(\exp\left(\frac{\xi}{t}\right)^\gamma\right) \psi(t, x)\, dt^{-\gamma}. \tag{1.4}$$

Let $\phi_i(\xi, x) \in \mathcal{O}(S^*_{\{0\}} \times U)(i = 1, 2)$ satisfying $|\phi_i(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma}$. Then γ-convolution of $\phi_1(\xi, x)$ and $\phi_2(\xi, x)$ is defined by

$$(\phi_1 \underset{\gamma}{*} \phi_2)(\xi, x) = \int_0^\xi \phi_1((\xi^\gamma - \eta^\gamma)^{1/\gamma}) \phi_2(\eta)\, d\eta^\gamma, \qquad \xi \in S^*_{\{0\}}. \tag{1.5}$$

The following relations hold between γ-Laplace transform, γ-Borel transform and γ-convolution.

LEMMA 1.2. *Suppose that* $\phi_i(\xi, x) \in \mathrm{Exp}(\gamma, S^* \times U)$ $(i = 0, 1, 2)$ *satisfy* $|\phi_i(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma}$ $(\varepsilon > 0)$ *in* $\{(\xi, x) \in S^* \times U;\ 0 < |\xi| < \rho_0\}$. *Then*

$$\mathcal{B}_{\gamma,\theta}\mathcal{L}_{\gamma,\theta}\phi_0 = \phi_0, \tag{1.6}$$

$$(\mathcal{L}_{\gamma,\theta}\phi_1)(\mathcal{L}_{\gamma,\theta}\phi_2) = \mathcal{L}_{\gamma,\theta}(\phi_1 \underset{\gamma}{*} \phi_2). \tag{1.7}$$

LEMMA 1.3. *Let $s, s_1, s_2 > 0$. Then*

$$\int_0^{\infty} (\exp - (\xi/t)^{\gamma}) \frac{\xi^{s-\gamma}}{\Gamma(s/\gamma)} \, d\xi^{\gamma} = t^s, \tag{1.8}$$

$$\xi^{s_1-\gamma} \underset{\gamma}{*} \xi^{s_2-\gamma} = \frac{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)}{\Gamma((s_1+s_2)/\gamma)} \xi^{s_1+s_2-\gamma}. \tag{1.9}$$

LEMMA 1.4. *Let $0 < \gamma \leqslant \kappa$. Suppose that $\phi_i(\xi, x) \in \mathcal{O}(S^* \times U)$ $(i = 1, 2)$ satisfy*

$$|\phi_i(\xi, x)| \leqslant \frac{C_i |\xi|^{s_i-\gamma} e^{c|\xi|^{\kappa}}}{\Gamma(s_i/\gamma)} \qquad (s_i > 0) \quad \text{for } (\xi, x) \in S^* \times U.$$

Then $(\phi_1 \underset{\gamma}{*} \phi_2)(\xi, x) \in \mathcal{O}(S^* \times U)$ and

$$|(\phi_1 \underset{\gamma}{*} \phi_2)(\xi, x)| \leqslant \frac{C_1 C_2 |\xi|^{s_1+s_2-\gamma} e^{c|\xi|^{\kappa}}}{\Gamma((s_1+s_2)/\gamma)}. \tag{1.10}$$

Proof. We have

$$\begin{aligned} (\phi_1 \underset{\gamma}{*} \phi_2)(\xi) &= \int_0^{|\xi| e^{i \arg \xi}} \phi_1((\xi^{\gamma} - \eta^{\gamma})^{1/\gamma}) \phi_2(\eta) \, d\eta^{\gamma} \\ &= \int_0^{|\xi|} \phi_1((|\xi|^{\gamma} - r^{\gamma})^{1/\gamma} e^{i \arg \xi}) \phi_2(r e^{i \arg \xi}) e^{i\gamma \arg \xi} \, dr^{\gamma}. \end{aligned}$$

Hence, by $(|\xi|^{\gamma} - r^{\gamma})^{\kappa/\gamma} + r^{\kappa} \leqslant |\xi|^{\kappa}$ for $0 \leqslant r \leqslant |\xi|$,

$$\begin{aligned} |(\phi_1 \underset{\gamma}{*} \phi_2)(\xi)| &\leqslant \int_0^{|\xi|} |\phi_1((|\xi|^{\gamma} - r^{\gamma})^{1/\gamma} e^{i \arg \xi}) \phi_2(r e^{i \arg \xi})| \, dr^{\gamma} \\ &\leqslant \frac{C_1 C_2}{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)} \int_0^{|\xi|} (|\xi|^{\gamma} - r^{\gamma})^{s_1/\gamma - 1} e^{c(|\xi|^{\gamma} - r^{\gamma})^{\kappa/\gamma}} r^{s_2-\gamma} e^{c r^{\kappa}} \, dr^{\gamma} \\ &\leqslant \frac{C_1 C_2 e^{c|\xi|^{\kappa}}}{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)} \int_0^{|\xi|} (|\xi|^{\gamma} - r^{\gamma})^{s_1/\gamma - 1} r^{s_2-\gamma} \, dr^{\gamma} \\ &\leqslant \frac{C_1 C_2 |\xi|^{s_1+s_2-\gamma} e^{c|\xi|^{\kappa}}}{\Gamma((s_1+s_2)/\gamma)}. \quad \blacksquare \end{aligned}$$

We define the (γ', γ)-acceleration in the direction θ denoted by $\mathscr{A}_{\gamma',\gamma,\theta}$, which is introduced by Ecalle. Let $0 < \gamma < \gamma'$ and $\kappa^{-1} = \gamma^{-1} - (\gamma')^{-1}$ and recall

$S^* = S^*(\theta, \delta)$. Set

$$\mathscr{A}_{\gamma',\gamma,\theta} := \mathscr{B}_{\gamma',\theta}\mathscr{L}_{\gamma,\theta}, \tag{1.11}$$

which acts on $\phi(\xi, x) \in \mathrm{Exp}(\gamma, S^* \times U)$ satisfying (1.2). It is shown by Ecalle that $\mathscr{A}_{\gamma',\gamma,\theta}$ can be extended to $\phi(\xi, x) \in \mathrm{Exp}(\kappa, S^* \times U)$ with (1.2) and $(\mathscr{A}_{\gamma',\gamma,\theta}\phi)(\xi, x)$ is holomorphic in $S^*_{\{0\}}(\theta, \delta + \pi/2\kappa) \times U$.

LEMMA 1.5. *Let* $\phi_i(\xi, x) \in \mathrm{Exp}(\kappa, S^* \times U)$ $(i = 1, 2)$ *with* (1.2). *Then*

$$(\mathscr{A}_{\gamma',\gamma,\theta}\phi_1) \underset{\gamma'}{*} (\mathscr{A}_{\gamma',\gamma,\theta}\phi_2) = \mathscr{A}_{\gamma',\gamma,\theta}(\phi_1 \underset{\gamma}{*} \phi_2). \tag{1.12}$$

Let us define the formal γ-Borel transform, the formal γ-Laplace transform, and the formal γ-convolution for formal series. Let $\hat{v}(t, x) = \sum_{n=0}^{\infty} v_n(x)t^{n+a} \in t^a\mathcal{O}(U)[[t]]$, where $a \geqslant 0$. Then the formal γ-Borel transform $\hat{\mathscr{B}}_\gamma \hat{v}$ is defined by

$$(\hat{\mathscr{B}}_\gamma \hat{v})(\xi, x) := \begin{cases} v_0(x)\delta(\xi) + \sum\limits_{n=1}^{\infty} \dfrac{v_n(x)\xi^{n-\gamma}}{\Gamma\left(\frac{n}{\gamma}\right)} & \text{for } a = 0, \\ \sum\limits_{n=0}^{\infty} \dfrac{v_n(x)\xi^{n+a-\gamma}}{\Gamma\left(\frac{n+a}{\gamma}\right)} & \text{for } a > 0, \end{cases} \tag{1.13}$$

where $\delta(\xi)$ means the delta function with support at $\xi = 0$. In the following the notation $\xi^{-\gamma}/\Gamma(0/\gamma)$ means $\delta(\xi)$.

Let $\hat{v}^{\,*}(\xi, x) = \sum\limits_{n=0}^{\infty} v_n(x)\xi^{n+a-\gamma}/\Gamma\left(\frac{n+a}{\gamma}\right)$ $(a \geqslant 0)$ be a formal series in ξ. Then the formal γ-Laplace transform $\hat{\mathscr{L}}_\gamma \hat{v}^{\,*}$ is defined by

$$(\hat{\mathscr{L}}_\gamma \hat{v}^{\,*})(t, x) := \sum_{n=0}^{\infty} v_n(x)t^{n+a} \tag{1.14}$$

and it holds that

$$\hat{\mathscr{L}}_\gamma \hat{\mathscr{B}}_\gamma \hat{v} = \hat{v}, \qquad \hat{\mathscr{B}}_\gamma \hat{\mathscr{L}}_\gamma \hat{v}^{\,*} = \hat{v}^{\,*} \tag{1.15}$$

and the formal (γ', γ)-acceleration is defined by $\hat{\mathscr{A}}_{\gamma',\gamma} := \hat{\mathscr{B}}_{\gamma'}\hat{\mathscr{L}}_\gamma$ for $0 < \gamma < \gamma'$. Finally, we define the formal γ-convolution. Let $\hat{v}_i^{\,*}(\xi, x) = \sum_{n=0}^{\infty} v_{i,n}(x)$ $\xi^{n+a_i-\gamma}/\Gamma\left(\frac{n+a_i}{\gamma}\right)$, $a_i \geqslant 0$, $i = 1, 2$. Then the formal γ-convolution

$(\hat{v}_1^* \underset{\gamma}{*} \hat{v}_2^*)(\xi, x)$ is define by

$$(\hat{v}_1^* \underset{\gamma}{*} \hat{v}_2^*)(\xi, x) := \sum_{n=0}^{\infty} \frac{c_n(x)\xi^{n+a_1+a_2-\gamma}}{\Gamma(\frac{n+a_1+a_2}{\gamma})},$$
$$c_n(x) = \sum_{l+m=n} v_{1,l}(x)v_{2,m}(x). \tag{1.16}$$

Remark. 1.6. We have for any smooth function $\varphi(\xi)$ with compact support

$$\lim_{s\to+0}\int_0^{\infty} \frac{\xi^{s-\gamma}}{\Gamma(s/\gamma)}\,\varphi(\xi)\,d\xi^{\gamma} = \varphi(0).$$

So it is reasonable that the notation $\frac{\xi^{-\gamma}}{\Gamma(0/\gamma)}$ means $\delta(\xi)$. By the definition, $\frac{\xi^{-\gamma}}{\Gamma(0/\gamma)} \underset{\gamma}{*} v^*(\xi, x) = v^*(\xi, x)$ and $\hat{\mathscr{A}}_{\gamma',\gamma}\frac{\xi^{-\gamma}}{\Gamma(0/\gamma)} = \frac{\xi^{-\gamma'}}{\Gamma(0/\gamma')}$.

We have

LEMMA 1.7. *Let* $\hat{v}_i(t, x) = \sum_{n=0}^{\infty} v_{i,n}(x)t^{n+a_i} \in t^{a_i}\mathcal{O}(U)[[t]]$ $(a_i \geqslant 0)$ $(i = 1, 2)$. *Then*

$$(\hat{\mathscr{B}}_\gamma(\hat{v}_1\hat{v}_2))(\xi, x) = ((\hat{\mathscr{B}}_\gamma\hat{v}_1) \underset{\gamma}{*} (\hat{\mathscr{B}}_\gamma\hat{v}_2))(\xi, x). \tag{1.17}$$

The proof is easy.

LEMMA 1.8. *Let* $\hat{v}(t, x) = \sum_{n=1}^{\infty} v_n(x)t^{n+a} \in t^{1+a}\mathcal{O}(U)[[t]]$ $(a \geqslant 0)$ *and* $\hat{b}(t, x) = \sum_{n=0}^{\infty} b_n(x)t^{n+b} \in t^b\mathcal{O}(U)[[t]]$ $(b \geqslant 0)$. *Suppose that* $(\hat{\mathscr{B}}_\gamma\hat{v})(\xi, x)$ *and* $(\hat{\mathscr{B}}_\gamma\hat{b}_1)(\xi, x)$, *where* $\hat{b}_1(t, x) = \hat{b}(t, x) - b_0(x)t^b$, *converge in* $\{(\xi, x); 0 < |\xi| < \rho_0, x \in U\}$. *Then* $(\hat{\mathscr{B}}_\gamma(\hat{b}\hat{v}))(\xi, x)$ *converges and*

$$(\hat{\mathscr{B}}_\gamma(\hat{b}\hat{v}))(\xi, x) = ((\hat{\mathscr{B}}_\gamma\hat{b}) \underset{\gamma}{*} (\hat{\mathscr{B}}_\gamma\hat{v}))(\xi, x) \tag{1.18}$$

holds in $\{(\xi, x); 0 < |\xi| < \rho_0, x \in U\}$.

Proof. The right-hand side of (1.18) has analytical mean, that is, if $b > 0$,

$$\begin{aligned}(\hat{\mathscr{B}}_\gamma \hat{b}) \underset{\gamma}{*} (\hat{\mathscr{B}}_\gamma \hat{v})(\xi, x) &= \int_0^\xi (\hat{\mathscr{B}}_\gamma \hat{b})((\xi^\gamma - \eta^\gamma)^{1/\gamma})(\hat{\mathscr{B}}_\gamma \hat{v})(\eta, x)\, d\eta^\gamma \\ &= \sum_{n=1}^\infty \sum_{l+m=n} \frac{b_l(x) v_m(x)}{\Gamma((l+b)/\gamma)\Gamma((m+a)/\gamma)} \\ &\quad \times \int_0^\xi (\xi^\gamma - \eta^\gamma)^{(l+b)/\gamma - 1} \eta^{m+a-\gamma}\, d\eta^\gamma \\ &= \sum_{n=1}^\infty \left(\sum_{l+m=n} b_l(x) v_m(x) \right) \frac{\xi^{n+a+b-\gamma}}{\Gamma((n+a+b)/\gamma)}\end{aligned}$$

for $0 < |\xi| < \rho_0$ and (1.18) is analytically valid. If $b = 0$,

$$\begin{aligned}(\hat{\mathscr{B}}_\gamma \hat{b}) \underset{\gamma}{*} (\hat{\mathscr{B}}_\gamma \hat{v})(\xi, x) = b_0(x)(\mathscr{B}_\gamma \hat{v})(\xi, x) + \int_0^\xi (\hat{\mathscr{B}}_\gamma \hat{b}_1)((\xi^\gamma - \eta^\gamma)^{1/\gamma}) \\ \times (\hat{\mathscr{B}}_\gamma \hat{v})(\eta, x)\, d\eta^\gamma\end{aligned}$$

and we have (1.18). ∎

Finally, let us define multisummability of $\hat{f}(t, x) = \sum_{n=1}^\infty f_n(x) t^n \in t\mathcal{O}(U)[[t]]$. Let $0 < k_r < k_{r-1} < \cdots < k_1 < k_0 = +\infty$ and define κ_i by $\kappa_i^{-1} = k_i^{-1} - k_{i-1}^{-1}$ for $1 \leqslant i \leqslant r$. Let $S_i := S(\theta_i, \pi/2k_i + \varepsilon_i)$, $\varepsilon_i > 0$, $(1 \leqslant i \leqslant r)$ be sectors such that $S_{i-1} \subset S_i$. Set $\mathbf{k} = (k_1, \ldots, k_r)$, $\mathbf{S} = (S_1, \ldots, S_r)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)$. We call $\mathbf{S}$ a multisector and $\boldsymbol{\theta}$ a multidirection.

Then $\hat{f}(t, x) \in t\mathcal{O}(U)[[t]]$ is $\mathbf{k}$-summable ($(k_1, k_2, \ldots, k_r)$-summable) in multisector $\mathbf{S}$ (or multidirection $\boldsymbol{\theta}$), if the following conditions are satisfied:

(1) Let $f^r(\xi, x) := (\hat{\mathscr{B}}_{k_r} \hat{f})(\xi, x)$. Then there is a $\rho_0 > 0$ such that $f^r(\xi, x)$ converges uniformly on any compact set in $\{0 < |\xi| < \rho_0\} \times U$.

(2) Let $i \in \{1, 2, \ldots, r-1, r\}$. $f^i(\xi, x)$ can be holomorphically extensible to $S_i^* := S^*(\theta_i, \varepsilon_i) \times U$ and is of exponential growth of order κ_i, that is, there exist constants A and c such that

$$|f^i(\xi, x)| \leqslant A \exp(c|\xi|^{\kappa_i}) \qquad \text{on} \qquad (S_i^* \cap \{|\xi| \geqslant 1\}) \times U, \tag{1.19}$$

and if $i \neq 1$, define $f^{i-1}(\xi, x) := (\mathscr{A}_{k_{i-1}, k_i, \theta_i} f^i)(\xi, x)$, which is holomorphic in $S^*_{\{0\}}(\theta_i, \pi/2\kappa_i + \varepsilon_i) \times U$.

Then $\mathbf{k}$-sum of $\hat{f}(t, x)$ in multisector $\mathbf{S}$ (multidirection $\boldsymbol{\theta}$) is defined by $(\mathscr{L}_{k_1, \theta_1} f^1)(t, x)$ and denoted by simply $f(t, x)$. We have, by considering the behavior at $\xi = 0$ and (1.19), for any polydisk $V \Subset U$

$$|f^i(\xi, x)| \leqslant A|\xi|^{1-k_i} \exp(c|\xi|^{\kappa_i}) \quad \text{on } S_i^* \times V. \tag{1.20}$$

For $\hat{g}(t, x) = \sum_{n=0}^{\infty} g_n(x)t^n \in \mathcal{O}(U)[[t]]$, if $\hat{f}(t, x) := (\hat{g}(t, x) - g_0(x)) \in t\mathcal{O}(U)[[t]]$ is **k**-summable in multisector **S**, we say that $\hat{g}(t, x)$ is **k**-summable in multisector **S**, define its **k**-sum by $g_0(x) + (\mathscr{L}_{k_1,\theta_1} f^1)(t, x)$ and denote it simply by $g(t, x)$. $g(t, x)$ is holmorphic in $S_{1,\{0\}} \times U$ and $g(t, x) \sim \hat{g}(t, x)$ as $t \to 0$ in $S_{1,\{0\}} \times U$. Set $g^i(\xi, x) = g_0(x)\frac{\xi^{-k_i}}{\Gamma(0/k_i)} + f^i(\xi, x)$ for $1 \leqslant i \leqslant r$ (see Remark 1.6).

Lemma 1.9. *Let $\hat{g}(t, x) = \sum_{n=0}^{\infty} g_n(x)t^n \in \mathcal{O}(U)[[t]]$ be* **k***-summable in multisector* **S** *and $\delta > 0$. Let V be a polydisk centered at $x = 0$ such that $V \Subset U$. Then there are constants C_0 and c_0 such that*

$$\left| \frac{\xi^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} \underset{k_i}{*} g^i(\xi, x) \right| \leqslant \frac{C_0|\xi|^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} \exp(c_0|\xi|^{\kappa_i}) \quad \text{on } S_i^* \times V. \tag{1.21}$$

C_0 and c_0 are independent of δ.

Proof. Set $\hat{f}(t, x) := (\hat{g}(t, x) - g_0(x)) \in t\mathcal{O}(U)[[t]]$. Then

$$\frac{\xi^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} \underset{k_i}{*} g^i(\xi, x) = g_0(x)\frac{\xi^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} + \frac{\xi^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} \underset{k_i}{*} f^i(\xi, x).$$

So, by Lemma 1.4 and (1.20),

$$\begin{aligned} \left| \frac{\xi^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} \underset{k_i}{*} g^i(\xi, x) \right| &\leqslant C\left(\frac{|\xi|^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} + \frac{|\xi|^{\delta+1-k_i}}{\Gamma\left(\frac{\delta+1}{k_i}\right)} \exp(c|\xi|^{\kappa_i}) \right) \\ &\leqslant C_0 \frac{|\xi|^{\delta-k_i}}{\Gamma\left(\frac{\delta}{k_i}\right)} \exp(c_0|\xi|^{\kappa_i}). \quad \blacksquare \end{aligned}$$

2. CHARACTERISTIC POLYGON AND MULTISUMMABILITY OF SOLUTIONS

Let $U = \{x \in \mathbb{C}^d; |x| < R'\}$ be a polydisk and $\hat{P}(t, x, \partial_t, \partial_x)$ be a formal partial differential operator with coefficients in $\mathcal{O}(U)[[t]]$ with order m

$$\hat{P}(t, x, \partial_t, \partial_x) = \sum_{|\alpha| \leqslant m} t^{e_{\hat{P},\alpha}} \hat{c}_\alpha(t, x)(t\partial_t)^{\alpha_0} \partial_x^{\alpha'}, \quad e_{\hat{P},\alpha} \in \mathbb{N}, \tag{2.1}$$

where $\hat{c}_\alpha(t,x) \in \mathcal{O}(U)[[t]]$ and $\hat{c}_\alpha(0,x) \not\equiv 0$ for $\hat{c}_\alpha(t,x) \not\equiv 0$. Here we define the characteristic polygon (Newton polygon) of $\hat{P}(t,x,\partial_t,\partial_x)$ with respect to $\{t=0\}$ for our purpose. We refer characteristic polygons for linear partial differential operators to [8]. We introduce a new notation $\lrcorner(a,b)$ which means an infinite rectangle with lower right corner (a,b), $\lrcorner(a,b) := \{(x,y) \in \mathbb{R}^2; x \leqslant a, y \geqslant b\}$. The characteristic polygon Σ is defined by *the convex hull of* $\bigcup_\alpha \lrcorner(|\alpha|, e_{\hat{P},\alpha})$. The boundary of Σ consists of a vertical half-line $\Sigma(0)$, segments $\Sigma(i)$ $(1 \leqslant i \leqslant p^* - 1)$ and a horizontal half-line $\Sigma(p^*)$. Let γ_i be the slope of $\Sigma(i)$. Then $0 = \gamma_{p^*} < \gamma_{p^*-1} < \cdots < \gamma_1 < \gamma_0 = +\infty$. We call $\{\gamma_i\}_{0 \leqslant i \leqslant p^*}$ the characteristic indices of $\hat{P}(t,x,\partial_t,\partial_x)$ with respect to $\{t=0\}$. Let $\{(m_i, e(i)) \in \mathbb{R}^2; 0 \leqslant i \leqslant p^* - 1\}$ be the set of vertices of Σ, $0 \leqslant m_{p^*-1} < \cdots < m_1 < m_0 = m$, and set $m_{p^*} = 0$. The endpoints of the segment $\Sigma(i)$ are $(m_{i-1}, e(i-1))$ and $(m_i, e(i))$ (Fig. 1). Introduce the subset $\mathcal{I}_i = \{\alpha \in \mathbb{N}^{d+1}; (|\alpha|, e_{\hat{P},\alpha}) \in \Sigma(i)\}$ of the multiindices and set

$$A_i(t,x,\partial_t,\partial_x) = \sum_{\alpha \in \mathcal{I}_i} t^{e_{\hat{P},\alpha}} \hat{c}_\alpha(0,x)(t\partial_t)^{\alpha_0} \partial_x^{\alpha'}. \tag{2.2}$$

It follows from the definition of the characteristic polygon Σ that

$$e_{\hat{P},\alpha} \geqslant e(p^* - 1),$$

$$e_{\hat{P},\alpha} - e(i) \geqslant \gamma_i(|\alpha| - m_i) \tag{2.3}$$

and $e_{\hat{P},\alpha} - e(i) = \gamma_i(|\alpha| - m_i)$ if and only if $\alpha \in \mathcal{I}_i$. We give a lemma for the later purposes (Fig. 1).

LEMMA 2.1. *Suppose $p^* \geqslant 2$ and let $1 \leqslant i \leqslant p^* - 1$. Then*

$$\gamma_i m_i - e(i) < \gamma_{i-1} m_{i-1} - e(i-1). \tag{2.4}$$

Further if $e(p^* - 1) = 0$, then $\gamma_i m_i - e(i) > 0$ for $i \neq p^* - 1$.

Proof. By the definition of γ_i we have $\gamma_i m_i - e(i) = \gamma_i m_{i-1} - e(i-1) < \gamma_{i-1} m_{i-1} - e(i-1)$. So $\gamma_i m_i - e(i) > \gamma_{p^*-1} m_{p^*-1} - e(p^* - 1)$ for $i \neq p^* - 1$ and we have the last assertion. ∎

We give several conditions on $\hat{P}(t,x,\partial_t,\partial_x)$.
Condition 0. If $\alpha = (\alpha_0, \alpha') \in \bigcup_{i=1}^{p^*} \mathcal{I}_i$, then $|\alpha'| = 0$.

If $\hat{P}(t,x,\partial_t,\partial_x)$ satisfies Condition 0, then $A_i(t,x,\partial_t,\partial_x)$ is an ordinary differential operator for all $1 \leqslant i \leqslant p^*$, so we denote it by $A_i(t,x,\partial_t)$ and

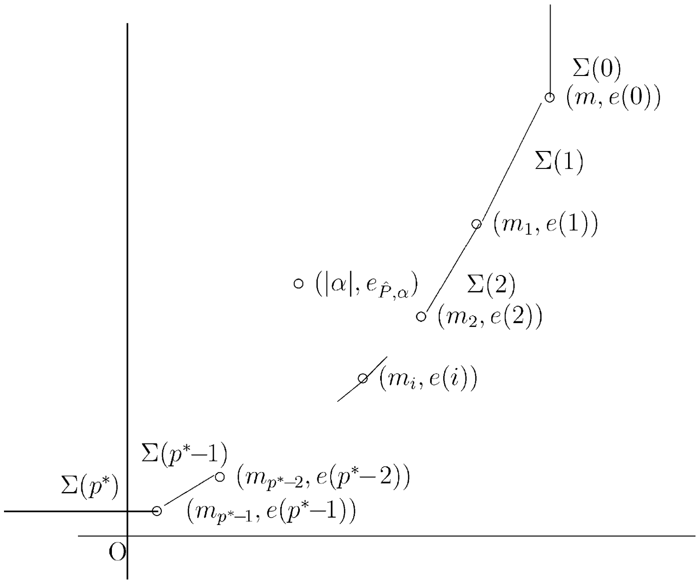

FIG. 1. Characteristic polygon.

represent it in another form

$$A_i(t, x, \partial_t) = \sum_{(h,0')\in \mathscr{I}_i} t^{e_h} a_h(x)(t\partial_t)^h, \tag{2.5}$$

where $(h, 0') = (h, 0, \ldots, 0) \in \mathbb{N} \times \mathbb{N}^d$, $e_h = e_{\hat{P},(h,0')}$ and $a_h(x) = \hat{c}_{(h,0')}(0, x)$. It holds that $e_h - e(i) = \gamma_i(h - m_i)$ for $(h, e_h) \in \mathscr{I}_i$.

Condition 1. Under Condition 0

$$a_{m_i}(0) \neq 0 \qquad \text{for all } 0 \leqslant i \leqslant p^* - 1. \tag{2.6}$$

Suppose Condition 1 holds. Then there is $R > 0, (0 < R < R')$ such that for all $0 \leqslant i \leqslant p^* - 1$

$$a_{m_i}(x) \neq 0 \qquad \text{on } \{|x| \leqslant R\}. \tag{2.7}$$

Set for $1 \leqslant i \leqslant p^* - 1$

$$\begin{cases} A_i(\xi, x) = \sum_{(h,0')\in \mathscr{I}_i} a_h(x)(\gamma_i \xi)^h, \\ A_i^0(\xi, x) = \sum_{(h,0')\in \mathscr{I}_i} a_h(x)(\gamma_i \xi)^{h-m_i}. \end{cases} \tag{2.8}$$

Then $A_i(\xi, x) = (\gamma_i\xi)^{m_i} A_i^0(\xi, x)$ and $A_i^0(\xi, x)$ is a polynomial in ξ with degree $(m_{i-1} - m_i)$ and $A_i^0(0, x) = a_{m_i}(x) \neq 0$ on $\{|x| \leqslant R\}$.

Definition 2.2. Suppose Condition 1 holds and $p^* \geqslant 2$. Let $i \in \{1, 2, \dots, p^* - 1\}$. Set $Z_i(r) = \bigcup_{|x|\leqslant r} \{\xi; A_i^0(\xi^{\gamma_i}, x) = 0\}$. A singular direction of level γ_i on $\{|x| \leqslant r\}$ is an argument of an element of $Z_i(r)$. We denote by $\Xi_i(r)$ the totality of singular directions on $\{|x| \leqslant r\}$ of level γ_i.

Condition 2. $p^* \geqslant 2$ and set $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_{p^*-1})$. The coefficients $\hat{c}_\alpha(t, x)$ of $\hat{P}(t, x, \partial_t, \partial_x)$ and $\hat{f}(t.x) \in \mathcal{O}(U)[[t]]$ are $\boldsymbol{\gamma}$-summable on a multisector $\mathbf{S} = (S_1, \dots, S_{p^*-1})$. The $\boldsymbol{\gamma}$-sum of $\hat{c}_\alpha(t, x)$ ($\hat{f}(t.x)$) is denoted by $c_\alpha(t, x)$ (resp. $f(t, x)$).

Assume Condition 2. Then there are a multidirection $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{p^*-1})$ and a multisector $\mathbf{S} = (S_1, \dots, S_{p^*-1})$ such that $S_i := S(\theta_i, \pi/2\gamma_i + \varepsilon_i)$ ($\varepsilon_i > 0$) with $S_{i-1} \subset S_i$, and

$$P(t, x, \partial_t, \partial_x) = \sum_{|\alpha|\leqslant m} t^{e_{P,\alpha}} c_\alpha(t, x)(t\partial_t)^{\alpha_0} \partial_x^{\alpha'} \tag{2.9}$$

is a linear partial differential operator with $\boldsymbol{\gamma}$-summable coefficients, where $e_{P,\alpha} := e_{\hat{P},\alpha}$.

We decompose $P(t, x, \partial_t, \partial_x)$ into two parts for our purpose. Set $\mathcal{I} = \bigcup_{i=1}^{p^*} \mathcal{I}_i$ and define

$$\begin{cases} A(t, x, \partial_t) = \sum_{(h,0')\in\mathcal{I}} t^{e_h} a_h(x)(t\partial_t)^h, \\ B(t, x, \partial_t, \partial_x) = P(t, x, \partial_t, \partial_x) - A(t, x, \partial_t). \end{cases} \tag{2.10}$$

It is obvious that the characteristic polygon of $A(t, x, \partial_t)$ is equal to that of $P(t, x, \partial_t, \partial_x)$. We can represent $B(t, x, \partial_t, \partial_x)$ in the following form:

$$B(t, x, \partial_t, \partial_x) = \sum_{\alpha} t^{e_{B,\alpha}} b_\alpha(t, x)(t\partial_t)^{\alpha_0} \partial_x^{\alpha'}, \tag{2.11}$$

where the coefficients $b_\alpha(t, x)$ ($\alpha \in \mathbb{N}^{d+1}$) are $\boldsymbol{\gamma}$-summable and $e_{B,\alpha} \in \mathbb{N}$ such that

$$e_{B,\alpha} - e(i) > \gamma_i(|\alpha| - m_i) \qquad \text{for all } 1 \leqslant i \leqslant p^*. \tag{2.12}$$

Now consider

$$P(t, x, \partial_t, \partial_x)u(t, x) = f(t, x). \tag{EQ}$$

The main result is

THEOREM 2.3. *Suppose that Conditions* 1 *and* 2 *hold. Let* $\mathbf{S}' = (S'_1, \dots, S'_{p^*-1})$ *be a multisector, where* $\underline{S'_i} = S(\theta_i, \pi/2\gamma_i + \varepsilon'_i)$ $(0 < \varepsilon'_i < \varepsilon_i)$ *and* $S'_{i-1} \subset S'_i$. *Set* $S'^*_i = S^*(\theta_i, \varepsilon'_i)$. *Suppose* $\overline{S'^*_i} \cap \overline{Z_i(R)} = \emptyset$ *for all* $1 \leqslant i \leqslant p^* - 1$. *Let* $\hat{u}(t, x) \in \mathcal{O}(U)[[t]]$ *be a formal solution of* (EQ). *Then there is a polydisk* $U' \subset U$ *such that* $\hat{u}(t, x) \in \mathcal{O}(U')[[t]]$ *is* γ*-summable on* $\mathbf{S}'$.

We note that S'_i is a proper subsector of S_i and $\overline{S'^*_i} \cap \overline{Z_i(R)} = \emptyset$ means $[\theta_i - \varepsilon'_i, \theta_i + \varepsilon'_i] \cap \overline{\Xi_i(R)} = \emptyset$. We assume $p^* \geqslant 2$ in Theorem 2.3. If $P(t, x, \partial_t, \partial_x)$ with holomorphic coefficients satisfies $p^* = 1$ and Condition 0, then $t^{-e(p^*-1)} P(t, x, \partial_t, \partial_x)$ is a partial differential operator of Fuchsian type with weight 0 with respect to $t = 0$ in the sense of Baouendi and Goulaouic [3], so every formal solution $\hat{u}(t, x) \in \mathcal{O}(U)[[t]]$ is convergent, provided $f(t, x)$ is holomorphic.

3. FORMAL EQUATIONS

Let

$$\begin{cases} \hat{P}(t, x, \partial_t, \partial_x) = \sum_{|\alpha| \leqslant m} t^{e_\alpha} \hat{c}_\alpha(t, x)(t\partial_t)^{\alpha_0} \partial_x^{\alpha'}, & e_\alpha \in \mathbb{N}, \\ \hat{P}(t, x, \partial_t, \partial_x)\hat{u}(t, x) = \hat{f}(t, x) \in t\mathcal{O}(U)[[t]] \end{cases} \tag{3.1}$$

be a formal partial differential equations. Suppose that (3.1) has a formal solution $\hat{u}(t, x) \in t\mathcal{O}(U)[[t]]$. The purpose of this section is to find a formal convolution equation that $(\hat{\mathcal{B}}_\gamma \hat{u})(\xi, x)$ satisfies. First, we introduce constants $C_{\gamma,k,s}$, which appear again in Lemma 4.1 in Section 4.

LEMMA 3.1. *Let* $C_{\gamma,k,s}$ $(1 \leqslant s \leqslant k)$ *be constants defined inductively by*

$$C_{\gamma,1,1} = \gamma, \quad C_{\gamma,k,s} = -\gamma s C_{\gamma,k-1,s} + \gamma C_{\gamma,k-1,s-1} \qquad \text{for } k \geqslant 2. \tag{3.2}$$

Then $C_{\gamma,k,k} = \gamma^k$ *and*

$$n^k = \sum_{s=1}^{k} C_{\gamma,k,s} \frac{\Gamma\left(\frac{n}{\gamma} + s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} \quad \text{for } n = 1, 2, \dots . \tag{3.3}$$

Proof. We show the lemma by induction on k. Let $k = 1$. Then $C_{\gamma,1,1} = \gamma$ and $n/\gamma = \Gamma\left(\frac{n}{\gamma} + 1\right)/\Gamma\left(\frac{n}{\gamma}\right)$, so (3.3) holds for $k = 1$. Let $k \geqslant 2$ and $C_{\gamma,k,s}$ be constants defined by (3.2). Then we have $C_{\gamma,k,k} = \gamma C_{\gamma,k-1,k-1} = \gamma^k$

and by (3.2)

$$\begin{aligned}
&\sum_{s=1}^{k} C_{\gamma,k,s}\frac{\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} \\
&= -\sum_{s=1}^{k-1}\frac{\gamma s C_{\gamma,k-1,s}\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} + \sum_{s=1}^{k-1}\frac{\gamma C_{\gamma,k-1,s}\Gamma\left(\frac{n}{\gamma}+s+1\right)}{\Gamma\left(\frac{n}{\gamma}\right)} \\
&= -\sum_{s=1}^{k-1}\frac{\gamma s C_{\gamma,k-1,s}\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} + \sum_{s=1}^{k-1}\frac{\left(\frac{n}{\gamma}+s\right)\gamma C_{\gamma,k-1,s}\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} \\
&= n\sum_{s=1}^{k-1} C_{\gamma,k-1,s}\frac{\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} = n^k. \quad \blacksquare
\end{aligned}$$

We define $C_{\gamma,k,0}$ as follows:

$$C_{\gamma,k,0} = \begin{cases} 1 & \text{for } k=0, \\ 0 & \text{for } k \geqslant 1. \end{cases} \tag{3.4}$$

Then

$$n^k = \sum_{s=0}^{k} C_{\gamma,k,s}\frac{\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)} \qquad \text{for } n=1,2,\dots\ . \tag{3.5}$$

The constants $C_{\gamma,k,s}$ appear in next lemma.

LEMMA 3.2. *Let* $\hat{v}(t,x)=\sum_{n=1}^{\infty} v_n(x)t^n \in t\mathcal{O}(U)[[t]]$ *and* $\phi(\xi,x)=(\hat{\mathscr{B}}_\gamma\hat{v})(\xi,x)$. *Then the formal* γ*-transform of* $t^\delta(t\partial_t)^k\hat{v}(t,x)$ $(\delta\geqslant\gamma k)$ *is given by*

$$(\hat{\mathscr{B}}_\gamma t^\delta(t\partial_t)^k\hat{v})(\xi,x) = \sum_{s=0}^{k} C_{\gamma,k,s}\frac{\xi^{\delta-\gamma(s+1)}}{\Gamma\left(\frac{\delta}{\gamma}-s\right)} \underset{\gamma}{*} (\xi^{\gamma s}\phi(\xi,x)). \tag{3.6}$$

Proof. It holds that $t^{\delta-\gamma s}\hat{\mathscr{L}}_\gamma(\xi^{\gamma s}\phi(\xi,x))=\sum_{n=1}^\infty v_n(x)t^{n+\delta}\frac{\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)}$ and its formal γ-Borel transform is $\frac{\xi^{\delta-\gamma(s+1)}}{\Gamma\left(\frac{\delta}{\gamma}-s\right)}\underset{\gamma}{*}(\xi^{\gamma s}\phi(\xi,x))$. By (3.5)

$$\begin{aligned} t^\delta(t\partial_t)^k\hat{v}(t,x) &= \sum_{n=1}^\infty v_n(x)n^k t^{n+\delta} \\ &= \sum_{n=1}^\infty v_n(x)\left(\sum_{s=0}^k C_{\gamma,k,s}\frac{\Gamma\left(\frac{n}{\gamma}+s\right)}{\Gamma\left(\frac{n}{\gamma}\right)}\right)t^{n+\delta} \\ &= \sum_{s=0}^k C_{\gamma,k,s}t^{\delta-\gamma s}\hat{\mathscr{L}}_\gamma(\xi^{\gamma s}\phi(\xi,x)). \end{aligned}$$

Therefore $(\mathscr{B}_\gamma t^\delta(t\partial_t)^k\hat{v})(\xi,x)=\sum_{s=0}^k C_{\gamma,k,s}\frac{\xi^{\delta-\gamma(s+1)}}{\Gamma\left(\frac{\delta}{\gamma}-s\right)}\underset{\gamma}{*}(\xi^{\gamma s}\phi(\xi,x))$. ∎

PROPOSITION 3.3. *Let* $\hat{v}(t,x)=\sum_{n=1}^\infty v_n(x)t^n\in t\mathcal{O}(U)[[t]]$, $\phi(\xi,x)=(\hat{\mathscr{B}}_\gamma\hat{v})(\xi,x)$ *and* $\hat{c}(t,x)=\sum_{n=0}^\infty c_n(x)t^n\in\mathcal{O}(U)[[t]]$. *Let* $k\in\mathbb{N}$ *and* δ *be a constant with* $\delta\geqslant\gamma k$. *Set* $\hat{c}_{k,s}(t,x)=C_{\gamma,k,s}t^{\delta-\gamma s}\hat{c}(t,x)$ *and* $c^*_{k,s}(\xi,x)=(\hat{\mathscr{B}}_\gamma\hat{c}_{k,s})(\xi,x)$ $(0\leqslant s\leqslant k)$. *Then*

$$(\hat{\mathscr{B}}_\gamma t^\delta\hat{c}(t,x)(t\partial_t)^k\hat{v})(\xi,x)=\sum_{s=0}^k c^*_{k,s}(\xi,x)\underset{\gamma}{*}(\xi^{\gamma s}\phi(\xi,x)). \tag{3.7}$$

Proof. By Lemma 3.2

$$\begin{aligned} (\hat{\mathscr{B}}_\gamma t^\delta\hat{c}(t,x)(t\partial_t)^k\hat{v}) &= ((\hat{\mathscr{B}}_\gamma\hat{c}(t,x))\underset{\gamma}{*}(\hat{\mathscr{B}}_\gamma t^\delta(t\partial_t)^k\hat{v}) \\ &= (\hat{\mathscr{B}}_\gamma\hat{c}(t,x))\underset{\gamma}{*}\left(\sum_{s=0}^k C_{\gamma,k,s}\frac{\xi^{\delta-\gamma(s+1)}}{\Gamma\left(\frac{\delta}{\gamma}-s\right)}\underset{\gamma}{*}(\xi^{\gamma s}\phi(\xi,x))\right) \\ &= \sum_{s=0}^k(\hat{\mathscr{B}}_\gamma\hat{c}_{k,s}(t,x))\underset{\gamma}{*}(\xi^{\gamma s}\phi(\xi,x)) \\ &= \sum_{s=0}^k c^*_{k,s}(\xi,x)\underset{\gamma}{*}(\xi^{\gamma s}\phi(\xi,x)). \quad ∎ \end{aligned}$$

Let us return to (3.1). Let $d \geqslant 0$ and consider

$$t^d \hat{P}(t, x, \partial_t, \partial_x)\hat{v}(t, x) = t^d \hat{f}(t, x). \tag{3.8}$$

Further assume $d \geqslant \max_\alpha \{-e_\alpha + \gamma\alpha_0\}$ and set $\hat{g}(t, x) = t^d \hat{f}(t, x)$. Define $\hat{c}_{\alpha,s}(t, x)$ as follows. For α with $\alpha_0 \geqslant 1$

$$\hat{c}_{\alpha,0}(t, x) = 0, \quad \hat{c}_{\alpha,s}(t, x) = C_{\gamma,k,s} t^{d+e_\alpha-\gamma s}\hat{c}_\alpha(t, x) \qquad \text{for } 1 \leqslant s \leqslant \alpha_0, \tag{3.9}$$

and $\hat{c}_{\alpha,0}(t, x) = t^{d+e_\alpha}\hat{c}_\alpha(t, x)$ for α with $\alpha_0 = 0$. Set

$$c^*_{\alpha,s}(\xi, x) = (\hat{\mathscr{B}}_\gamma \hat{c}_{\alpha,s})(\xi, x), \qquad g^*(\xi, x) = (\hat{\mathscr{B}}_\gamma \hat{g})(\xi, x) \tag{3.10}$$

and consider the formal convolution equation

$$\sum_\alpha \sum_{s=0}^{\alpha_0} c^*_{\alpha,s}(\xi, x) \underset{\gamma}{*} (\xi^{\gamma s}\phi(\xi, x)) = g^*(\xi, x). \tag{3.11}$$

It follows from the preceding arguments that Eq. (3.11) is much connected with (3.1) and (3.8). It is expected that the formal γ-Borel transform $(\hat{\mathscr{B}}_\gamma \hat{u})(\xi, x)$ of the formal solution $\hat{u}(t, x)$ of (3.1) is a solution of (3.11). Moreover, if the coefficients $c^*_{\alpha,s}(\xi, x)$ and $g^*(\xi, x)$ converge in $\{\xi; 0 < |\xi| < \rho_0\}$ for some $\rho_0 > 0$, then (3.11) is not formal but has analytical mean and $(\hat{\mathscr{B}}_\gamma \hat{u})(\xi, x)$ will be a genuine solution, provided it converges. We have

THEOREM 3.4. *Let $\hat{u}(t, x) = \sum_{n=1}^\infty u_n(x)t^n \in t\mathcal{O}(U)[[t]]$ be a formal solution of* (3.1). *Then $\phi(\xi, x) := (\hat{\mathscr{B}}_\gamma \hat{u})(\xi, x)$ is a solution of* (3.11). *Moreover, if the coefficients $c^*_{\alpha,s}(\xi, x)$, $g^*(\xi, x)$ and $\phi(\xi, x)$ converge in $\{\xi; 0 < |\xi| < \rho_0\}$, then $\phi(\xi, x)$ is a genuine solution.*

Proof. We have from Proposition 3.3, by putting $\delta = d + e_\alpha \geqslant \gamma\alpha_0$,

$$(\hat{\mathscr{B}}_\gamma t^{d+e_\alpha}\hat{c}_\alpha(t, x)(t\partial_t)^{\alpha_0}\partial_x^{\alpha'}\hat{u})(\xi, x) = \sum_{s=0}^{\alpha_0} c^*_{\alpha,s}(\xi, x) \underset{\gamma}{*} (\xi^{\gamma s}\partial_x^{\alpha'}\phi(\xi, x)).$$

Hence $\phi(\xi, x)$ satisfies

$$\begin{aligned} g^*(\xi, x) &= (\hat{\mathscr{B}}_\gamma t^d \hat{f})(\xi, x) = (\hat{\mathscr{B}}_\gamma t^d \hat{P}(t, x, \partial_t, \partial_x)\hat{u})(\xi, x) \\ &= \sum_\alpha \sum_{s=0}^{\alpha_0} c^*_{\alpha,s}(\xi, x) \underset{\gamma}{*} (\xi^{\gamma s}\partial_x^{\alpha'}\phi(\xi, x)). \end{aligned}$$

The last assertion is obvious. ∎

4. CONVOLUTION EQUATIONS

We have obtained a partial differential–convolution equation formally in Section 3. In this section we derive partial differential–convolution equations from (EQ) in Section 2 by the analytical method as was done in [4, 5], where convolution equations were used to show multisummability of formal solutions of ordinary equations.

Let U be a polydisk in $\mathbb{C}^d$ with center $x = 0$ and $\mathbb{R}_+ = (0, \infty)$. Let $\mathscr{C}_{\gamma,\theta}(U)$ be the totality of continuous functions $\phi(\xi, x)$ on $\{(\xi, x) \in e^{i\theta}\mathbb{R}_+ \times U\}$ which are holomorphic in $x \in U$, $\phi(\xi, x) = 0$ for $\xi \geqslant R$ for some $R > 0$ and $|\phi(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma}$ $(\varepsilon > 0)$ on $e^{i\theta}\mathbb{R}_+ \times U$. Set for $\phi(\xi, x) \in \mathscr{C}_{\gamma,\theta}(U)$

$$v(t, x) = \int_0^{\infty e^{i\theta}} \exp\left(-\left(\frac{\xi}{t}\right)^\gamma\right)\phi(\xi, x)\, d\xi^\gamma, \qquad \gamma > 0. \tag{4.1}$$

Firstly, we assume $\phi(\xi, x) \in \mathscr{C}_{\gamma,\theta}(U)$. However after we obtain equations, we will find that we may remove the assumption of the support of $\phi(\xi, x)$. We have

LEMMA 4.1. *Let $C_{\gamma,k,s}$ $(0 \leqslant s \leqslant k)$ be constants defined by* (3.2) *and* (3.4). *Then*

$$(t\partial_t)^k v(t, x) = \sum_{s=0}^{k} \frac{C_{\gamma,k,s}}{t^{\gamma s}} \int_0^{\infty e^{i\theta}} \xi^{\gamma s}(\exp - (\xi/t)^\gamma)\phi(\xi, x)\, d\xi^\gamma. \tag{4.2}$$

Proof. If $k = 0$, then $C_{\gamma,0,0} = 1$ and (4.2) is obvious. If $k = 1$, then $C_{\gamma,1,0} = 0$, $C_{\gamma,1,1} = \gamma$ and $t\partial_t v(t, x) = \gamma t^{-\gamma} \int_0^\infty \xi^\gamma(\exp(-(\xi/t)^\gamma)\phi(\xi, x)\, d\xi^\gamma$, so (4.2) holds for $k = 1$. Inductively,

$$\begin{aligned}
(t\partial_t)^k v(t, x) &= (t\partial_t)(t\partial_t)^{k-1} v(t, x) \\
&= (t\partial_t)\left(\sum_{s=0}^{k-1} \frac{C_{\gamma,k-1,s}}{t^{\gamma s}} \int_0^\infty \xi^{\gamma s}(\exp - (\xi/t)^\gamma)\phi(\xi, x)\, d\xi^\gamma\right) \\
&= -\sum_{s=0}^{k-1} \frac{\gamma s C_{\gamma,k-1,s}}{t^{\gamma s}} \int_0^\infty \xi^{\gamma s}(\exp - (\xi/t)^\gamma)\phi(\xi, x)\, d\xi^\gamma \\
&\quad + \sum_{s=0}^{k-1} \frac{\gamma C_{\gamma,k-1,s}}{t^{\gamma(s+1)}} \int_0^\infty \xi^{\gamma(s+1)}(\exp - (\xi/t)^\gamma)\phi(\xi, x)\, d\xi^\gamma \\
&= \sum_{s=0}^{k} \frac{C_{\gamma,k,s}}{t^{\gamma s}} \int_0^\infty \xi^{\gamma s}(\exp - (\xi/t)^\gamma)\phi(\xi, x)\, d\xi^\gamma \\
&\qquad \text{(by (3.2)).} \quad \blacksquare
\end{aligned}$$

Now let us return to (EQ), $P(t, x, \partial_t, \partial_x)u(t, x) = f(t, x)$, and suppose that Conditions 1 and 2 hold. We may assume, by multiplying $t^{-e(p^*-1)}$,

$$e(p^* - 1) = 0. \tag{4.3}$$

We summerize shortly what we need:

$$\begin{cases} P(t, x, \partial_t, \partial_x) = A(t, x, \partial_t) + B(t, x, \partial_t, \partial_x), \\ A(t, x, \partial_t) = \sum_{(h,0')\in\mathscr{I}} t^{e_h} a_h(x)(t\partial_t)^h, \\ B(t, x, \partial_t, \partial_x) = \sum_{\alpha} t^{e_{B,\alpha}} b_\alpha(t, x)(t\partial_t)^{\alpha_0}\partial_x^{\alpha'}. \end{cases} \tag{4.4}$$

Here $\mathscr{I} = \bigcup_{i=1}^{p^*} \mathscr{I}_i$, $\mathscr{I}_i = \{\alpha \in \mathbb{N}^{d+1}; (|\alpha|, e_\alpha) \in \Sigma(i)\}$, $b_\alpha(t, x)$ $(f(t, x))$ is the $\boldsymbol{\gamma}$-sum of $\hat{b}_\alpha(t, x)$ (resp. $\hat{f}(t, x)$). It holds that

$$e_{B,\alpha} - e(i) > \gamma_i(|\alpha| - m_i) \qquad \text{for all } 1 \leqslant i \leqslant p^* \tag{4.5}$$

and set $\kappa_i^{-1} = \gamma_i^{-1} - \gamma_{i-1}^{-1}$ for $1 \leqslant i \leqslant p^* - 1$. Since $b_\alpha(t, x)$ $(f(t, x))$ is the $\boldsymbol{\gamma}$-sum of $\hat{b}_\alpha(t, x)$ (resp. $\hat{f}(t, x)$) on a multisector $\mathbf{S} = (S_1, \ldots, S_{p^*-1})$, $S_i = S(\theta_i, \pi/2\gamma_i + \varepsilon_i)$, we use the notations $b_\alpha^i(\xi, x)$ and $f_\alpha^i(\xi, x)$ $(1 \leqslant i \leqslant p^* - 1)$:

$$\begin{cases} b_\alpha^{p^*-1}(\xi, x) = (\hat{\mathscr{B}}_{\gamma_{p^*-1}} \hat{b}_\alpha)(\xi, x), \quad b_\alpha^{i-1}(\xi, x) = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta_i} b_\alpha^i)(\xi, x), \\ f^{p^*-1}(\xi, x) = (\hat{\mathscr{B}}_{\gamma_{p^*-1}} \hat{f})(\xi, x), \quad f^{i-1}(\xi, x) = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta_i} f^i)(\xi, x). \end{cases} \tag{4.6}$$

We have $b_\alpha(t, x) = (\mathscr{L}_{\gamma_1,\theta_1} b_\alpha^1)(t, x)$ and $f(t, x) = (\mathscr{L}_{\gamma_1,\theta_1} f^1)(t, x)$.

Put $\gamma = \gamma_i$ and $\theta = \theta_i$ in (4.1). By Lemma 4.1 we have

$$\begin{aligned} t^{\gamma_i m_i - e(i)} A(t, x, \partial_t)v(t, x) &= \sum_{(h,0')\in\mathscr{I}} t^{\gamma_i m_i - e(i) + e_h} a_h(x)(t\partial_t)^h v(t, x) \\ &= \sum_{(h,0')\in\mathscr{I}} \sum_{s=0}^{h} a_{i,h,s}(t, x) \int_0^{\infty e^{i\theta_i}} \\ &\quad \times \xi^{\gamma_i s}\left(\exp - \left(\frac{\xi}{t}\right)^{\gamma_i}\right)\phi(\xi, x)\, d\xi^{\gamma_i}, \end{aligned}$$

where $a_{i,h,s}(t,x) = C_{\gamma_i,h,s} t^{\gamma_i m_i - e(i) + e_h - \gamma_i s} a_h(x)$, and

$$\begin{aligned} &t^{\gamma_i m_i - e(i)} B(t,x,\partial_t,\partial_x) v(t,x) \\ &\quad = \sum_{\alpha} t^{\gamma_i m_i - e(i) + e_{B,\alpha}} b_\alpha(t,x)(t\partial_t)^{\alpha_0} \partial_x^{\alpha'} v(t,x) \\ &\quad = \sum_{\alpha,s} b_{i,\alpha,s}(t,x) \int_0^{\infty e^{i\theta_i}} \xi^{\gamma_i s} \left(\exp - \left(\frac{\xi}{t}\right)^{\gamma_i}\right) \partial_x^{\alpha'} \phi(\xi,x)\, d\xi^{\gamma_i}, \end{aligned}$$

where $b_{i,\alpha,s}(t,x) = C_{\gamma_i,\alpha_0,s} t^{\gamma_i m_i - e(i) + e_{B,\alpha} - \gamma_i s} b_\alpha(t,x)$.

By considering $\mathscr{B}_{\gamma_i} t^s = \frac{\xi^{s-\gamma_i}}{\Gamma(s/\gamma_i)}$ and the definition of $b^i_\alpha(\xi,x)$, set

$$\begin{cases} a^i_{h,s}(\xi,x) = \dfrac{C_{\gamma_i,h,s}\xi^{e_h - e(i) + \gamma_i(m_i - s - 1)}}{\Gamma((e_h - e(i))/\gamma_i + m_i - s)} a_h(x), \\ b^i_{\alpha,s}(\xi,x) = \dfrac{C_{\gamma_i,\alpha_0,s}\xi^{e_{B,\alpha} - e(i) + \gamma_i(m_i - s - 1)}}{\Gamma((e_{B,\alpha} - e(i))/\gamma_i + m_i - s)} \underset{\gamma_i}{*} b^i_\alpha(\xi,x). \end{cases} \tag{4.7}$$

It holds by Lemma 1.9 that for $V \Subset U$

$$|b^i_{\alpha,s}(\xi,x)| \leqslant \frac{A|\xi|^{e_{B,\alpha} - e(i) + \gamma_i(m_i - s - 1)}}{\Gamma\left(\frac{e_{B,\alpha} - e(i)}{\gamma_i} + m_i - s\right)} \exp(c|\xi|^{\kappa_i}) \qquad \text{on } S^*_i(\theta_i,\varepsilon_i) \times V. \tag{4.8}$$

Thus, we can introduce following partial differential and convolution operators for $1 \leqslant i \leqslant p^* - 1$:

$$\mathfrak{P}_i(\xi,x,\partial_x) := \sum_{h,s} a^i_{h,s}(\xi,x) \underset{\gamma_i}{*} (\xi^{\gamma_i s}\cdot) + \sum_{\alpha,s} b^i_{\alpha,s}(\xi,x) \underset{\gamma_i}{*} (\xi^{\gamma_i s} \partial_x^{\alpha'} \cdot) \tag{4.9}$$

and let us consider

$$\begin{cases} \mathfrak{P}_i(\xi,x,\partial_x)\phi(\xi,x) = \psi^i(\xi,x), \\ \psi^i(\xi,x) = \dfrac{\xi^{\gamma_i m_i - e(i) - \gamma_i}}{\Gamma\left(m_i - \frac{e(i)}{\gamma_i}\right)} \underset{\gamma_i}{*} f^i(\xi,x). \end{cases} \tag{Eq $-$ i}$$

Here from the assumption $e(p^* - 1) = 0$, $\gamma_i m_i - e(i) > 0$ for $i \neq p^* - 1$ by Lemma 2.1. We derive (Eq-i) from $t^{\gamma_i m_i - e(i)} P(t,x,\partial_t,\partial_x) u(t,x) = t^{\gamma_i m_i - e(i)} f(t,x)$ by the restricting support of $\phi(\xi,x)$; however, it is obvious that Eq. (Eq-i) has meaning without the condition on support of $\phi(\xi,x)$.

Proposition 4.2. *Let* $\hat{u}(t,x) \in t\mathcal{O}(U)[[t]]$ *be a formal solution of* (EQ). *Set* $\phi(\xi,x) = (\hat{\mathscr{B}}_{\gamma_{p^*-1}} \hat{u})(\xi,x)$. *Then there is a constant* $\rho_0 > 0$ *and a* *neighborhood V of $x = 0$ such that $\phi(\xi, x) \in \mathcal{O}(\{\xi; 0 < |\xi| < \rho_0\} \times V)$ and is a solution of* (Eq-($p^* - 1$)).

In order to show Proposition 4.2 we need a lemma concerning the estimate of the coefficients of a formal solution.

LEMMA 4.3. *Let $\hat{u}(t, x) = \sum_{n=1}^{\infty} u_n(x)t^n$ be a formal solution of* (EQ). *Then there are constants A and B such that*

$$|u_n(x)| \leqslant AB^n \Gamma\left(\frac{n}{\gamma_{p^*-1}}\right) \tag{4.10}$$

in a neighborhood V of $x = 0$.

The proof of Lemma 4.3 is given in Section 6.

Proof of Proposition 4.2. By Lemma 4.3, $\phi(\xi, x)$ converges in $\{\xi; 0 < |\xi| < \rho_0\} \times V$, $\rho_0 = B^{-1}$. By putting $\gamma = \gamma_{p^*-1}$ and $d = \gamma_{p^*-1} m_{p^*-1}$, $\mathfrak{P}_{p^*-1}(\xi, x, \partial_x)$ is coincident with the formal partial differential operators obtained in Section 3. So it follows from Theorem 3.4 that $\phi(\xi, x)$ is a solution of (Eq-($p^* - 1$)). ∎

As for solutions of (Eq-i) ($1 \leqslant i \leqslant p^* - 1$), we have

PROPOSITION 4.4. *Let $i \in \{1, 2, \ldots, p^* - 1\}$, $S^* := S^*(\theta, \delta)$ be a sector with $\bar{S}^* \cap \bar{Z}_i(R) = \emptyset$ and $S^*_{\{0\}} = S^* \cap \{0 < |\xi| < \rho_0\}$. Let $\phi^i(\xi, x) \in \mathcal{O}(S^*_{\{0\}} \times V)$ be a solution of* (Eq-i). *Then $\phi^i(\xi, x)$ is holomorphically extensible to $S^* \times V'$ for some polydisk $V' \subset V$ centered at $x = 0$ and $\phi^i \in$* Exp*$(\kappa_i, S^* \times V')$.*

The proof of Proposition 4.4 is given in the following sections.

PROPOSITION 4.5. *Let $i \in \{2, 3, \ldots, p^* - 1\}$ and $S^* := S^*(\theta, \delta)$. Let $\phi^i(\xi, x) \in$* Exp*$(\kappa_i, S^* \times V)$ be a solution of* (Eq-i). *Then $\phi^{i-1}(\xi, x) := (\mathscr{A}_{\gamma_{i-1}, \gamma_i, \theta} \phi^i)(\xi, x)$ is a solution of* (Eq-($i - 1$)) *in $S^*_{\{0\}}(\theta, \pi/2\kappa_i + \delta) \times V$.*

We give lemmas to show Proposition 4.5.

LEMMA 4.6. *Let*

$$\chi_i^*(\xi) = \frac{\xi^{\gamma_{i-1} m_{i-1} - \gamma_i m_i + e(i) - e(i-1) - \gamma_i}}{\Gamma((\gamma_{i-1} m_{i-1} - \gamma_i m_i + e(i) - e(i-1))/\gamma_i)}. \tag{4.11}$$

Then

$$\psi^{i-1}(\xi, x) = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\psi_i). \tag{4.12}$$

Proof. By $(\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\psi_i) = \left(\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{\gamma_i m_i - e(i) - \gamma_i}}{\Gamma\left(m_i - \frac{e(i)}{\gamma_i}\right)}\right) \underset{\gamma_{i-1}}{*} f^{i-1}(\xi, x)$, we have

$$\begin{aligned}
&(\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{\gamma_i m_i - e(i) - \gamma_i}}{\Gamma\left(m_i - \frac{e(i)}{\gamma_i}\right)}\right) * f^{i-1}(\xi, x)\\
&\quad = \mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{\gamma_{i-1}m_{i-1} - e(i-1) - \gamma_i}}{\Gamma\left(\frac{\gamma_{i-1}m_{i-1} - e(i-1)}{\gamma_i}\right)} = \frac{\xi^{\gamma_{i-1}m_{i-1} - e(i-1) - \gamma_{i-1}}}{\Gamma\left(m_{i-1} - \frac{e(i-1)}{\gamma_{i-1}}\right)} * f^{i-1}(\xi, x)\\
&\quad = \psi^{i-1}(\xi, x). \quad \blacksquare
\end{aligned}$$

LEMMA 4.7. *Let $\phi(\xi, x) \in \mathrm{Exp}(\kappa_i, S^* \times V)$ and set $\Phi(\xi, x) = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\phi)(\xi, x)$. Let $k \in \mathbb{N}$ and ε be a constant with $\varepsilon \geqslant \max\{e(i-1) - \gamma_{i-1}(m_{i-1} - k), e(i) - \gamma_i(m_i - k)\}$. Then*

$$\begin{aligned}
&\sum_{s=0}^{k} \frac{C_{\gamma_{i-1},k,s}\xi^{\varepsilon - e(i-1) + \gamma_{i-1}(m_{i-1} - s - 1)})}{\Gamma((\varepsilon - e(i-1))/\gamma_{i-1} + m_{i-1} - s)} \underset{\gamma_{i-1}}{*} (\xi^{\gamma_{i-1}s}\Phi)\\
&\quad = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{k} C_{\gamma_i,k,s}\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{\varepsilon - e(i) + \gamma_i(m_i - s - 1)}}{\Gamma((\varepsilon - e(i))/\gamma_i + m_i - s)}\right.\\
&\qquad \left. \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\xi^{\gamma_i s}\phi)\right),
\end{aligned} \tag{4.13}$$

where $\chi_i^*(\xi)$ is the function defined by (4.11).

Proof. First, we assume $\phi(\xi, x) \in \mathrm{Exp}(\gamma_i, S^* \times V)$. Then it follows from Lemma 4.1 that

$$(t\partial_t)^k(\mathscr{L}_{\gamma_{i-1},\theta}\Phi)(t, x) = \sum_{s=0}^{k} C_{\gamma_{i-1},k,s}t^{-\gamma_{i-1}s}(\mathscr{L}_{\gamma_{i-1},\theta}\xi^{\gamma_{i-1}s}\Phi)(t, x),$$

$$(t\partial_t)^k(\mathscr{L}_{\gamma_i,\theta}\phi)(t, x) = \sum_{s=0}^{k} C_{\gamma_i,k,s}t^{-\gamma_i s}(\mathscr{L}_{\gamma_i,\theta}\xi^{\gamma_i s}\phi)(t, x).$$

Since $(\mathscr{L}_{\gamma_{i-1},\theta}\Phi)(t,x) = (\mathscr{L}_{\gamma_i,\theta}\phi)(t,x)$, we have from these equalities

$$\sum_{s=0}^{k} C_{\gamma_{i-1},k,s} t^{\varepsilon-e(i-1)+\gamma_{i-1}(m_{i-1}-s)}(\mathscr{L}_{\gamma_{i-1},\theta}\xi^{\gamma_{i-1}s}\Phi)(t,x)$$
$$= t^{\gamma_{i-1}m_{i-1}-\gamma_i m_i+e(i)-e(i-1)}\left(\sum_{s=0}^{k} C_{\gamma_i,k,s} t^{\varepsilon-e(i)+\gamma_i(m_i-s)}(\mathscr{L}_{\gamma_i,\theta}\xi^{\gamma_i s}\phi)(t,x)\right).$$

Hence, by putting $\chi_i(t) = t^{\gamma_{i-1}m_{i-1}-\gamma_i m_i+e(i)-e(i-1)}$,

$$\left(\sum_{s=0}^{k} \frac{C_{\gamma_{i-1},k,s}\xi^{\varepsilon-e(i-1)+\gamma_{i-1}(m_{i-1}-s-1)})}{\Gamma((\varepsilon-e(i-1))/\gamma_{i-1}+m_{i-1}-s)}\right) \underset{\gamma_{i-1}}{*} (\xi^{\gamma_{i-1}s}\Phi)$$
$$= (\mathscr{B}_{\gamma_{i-1},\theta}\chi_i) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{k} C_{\gamma_i,k,s}\mathscr{B}_{\gamma_{i-1},\theta} t^{\varepsilon-e(i)+\gamma_i(m_i-s)}\right) \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\xi^{\gamma_i s}\phi).$$

By $\mathscr{B}_{\gamma_{i-1},\theta}t^a = \frac{\xi^{a-\gamma_{i-1}}}{\Gamma(a/\gamma_{i-1})} = \mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{a-\gamma_i}}{\Gamma(a/\gamma_i)}$ and $\chi_i(t) = (\mathscr{L}_{\gamma_i,\theta}\chi_i^*)(t)$,

$$\left(\sum_{s=0}^{k} \frac{C_{\gamma_{i-1},k,s}\xi^{\varepsilon-e(i-1)+\gamma_{i-1}(m_{i-1}-s-1)})}{\Gamma((\varepsilon-e(i-1))/\gamma_{i-1}+m_{i-1}-s)}\right) \underset{\gamma_{i-1}}{*} (\xi^{\gamma_{i-1}s}\Phi)$$
$$= (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{k} C_{\gamma_i,k,s}\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{\varepsilon-e(i)+\gamma_i(m_i-s-1)}}{\Gamma((\varepsilon-e(i))/\gamma_i+m_i-s)}\right)$$
$$\underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\xi^{\gamma_i s}\phi).$$

Equality (4.13) holds for $\phi(\xi,x) \in \mathrm{Exp}(\kappa_i, S^* \times V)$ by the approximation. ∎

Proof of Proposition 4.5. Firstly, recall

$$\begin{cases} b_\alpha^{i-1}(\xi,x) = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}b_\alpha^i)(\xi,x), \\ b_{\alpha,s}^i(\xi,x) = \dfrac{C_{\gamma_i,\alpha_0,s}\xi^{e_{B,\alpha}-e(i)+\gamma_i(m_i-s-1)}}{\Gamma((e_{B,\alpha}-e(i))/\gamma_i+m_i-s)} \underset{\gamma_i}{*} b_\alpha^i(\xi,x), \end{cases}$$

where $(e_{B,\alpha}-e(i))/\gamma_i+m_i-s>0$. So we have

$$
\begin{aligned}
&\sum_{s=0}^{\alpha_0} b^{i-1}_{\alpha,s}(\xi,x) \underset{\gamma_{i-1}}{*} (\xi^{\gamma_{i-1}s}\phi^{i-1})\\
&= \left(\sum_{s=0}^{\alpha_0} \frac{C_{\gamma_{i-1},\alpha_0,s}\xi^{e_{B,\alpha}-e(i-1)+\gamma_{i-1}(m_{i-1}-s-1)}}{\Gamma((e_{B,\alpha}-e(i-1))/\gamma_{i-1}+m_{i-1}-s)}\right) \underset{\gamma_{i-1}}{*} b^{i-1}_{\alpha}(\xi,x) \underset{\gamma_{i-1}}{*} (\xi^{\gamma_{i-1}s}\phi^{i-1})\\
&= (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{\alpha_0} C_{\gamma_i,\alpha_0,s}\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\frac{\xi^{e_{B,\alpha}-e(i)+\gamma_i(m_i-s-1)}}{\Gamma((e_{B,\alpha}-e(i))/\gamma_i+m_i-s)}\right)\\
&\quad \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}b^i_\alpha) \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\xi^{\gamma_i s}\phi^i) \qquad \text{(by Lemma 4.7)}\\
&= (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{\alpha_0} \mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}b^i_{\alpha,s} \underset{\gamma_{i-1}}{*} \mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\xi^{\gamma_i s}\phi^i\right)\\
&= (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{\alpha_0} \mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}(b^i_{\alpha,s} \underset{\gamma_i}{*} (\xi^{\gamma_i s}\phi^i))\right)
\end{aligned}
$$

and by the same way

$$
\sum_{s=0}^{h} a^{i-1}_{h,s}(\xi,x) \underset{\gamma_{i-1}}{*} (\xi^{\gamma_{i-1}s}\phi_{i-1}) = (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} \left(\sum_{s=0}^{h} \mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}(a^i_{h,s} \underset{\gamma_i}{*} (\xi^{\gamma_i s}\phi_i))\right).
$$

Thus from (4.12)

$$
\begin{aligned}
\mathfrak{P}_{i-1}(\xi,x,\partial_x)\phi^{i-1}(\xi,x) &= (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\mathfrak{P}_i\phi^i)\\
&= (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\chi_i^*) \underset{\gamma_{i-1}}{*} (\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta}\psi^i) = \psi^{i-1}(\xi,x). \quad ∎
\end{aligned}
$$

Proof of Theorem 2.3. Let $\mathbf{S}'=(S'_1,\ldots,S'_{p^*-1})$, where $S'_i=S(\theta_i,\frac{\pi}{2\gamma_i}+\varepsilon'_i)$ and set $S'^*_i=S^*(\theta_i,\varepsilon'_i)$. Let $\hat{u}(t,x)\in\mathscr{O}(U)[[t]]$ be a formal solution of (EQ). We may assume $\hat{u}(t,x)\in t\mathscr{O}(U)[[t]]$ and set $\phi^{p^*-1}(\xi,x)=(\hat{\mathscr{B}}_{\gamma_{p^*-1}}\hat{u})(\xi,x)$. Then $\phi^{p^*-1}(\xi,x)$ is a solution of (Eq-(p^*-1)) by Proposition 4.2 and it is holomorphically extensible to $S'^*_{p^*-1}\times V_{p^*-1}$ and belongs to $\mathrm{Exp}(\kappa_{p^*-1},S'^*_{p^*-1}\times V_{p^*-1})$ by Proposition 4.4. So $\phi^{p^*-2}(\xi,x)=(\mathscr{A}_{\gamma_{p^*-2},\gamma_{p^*-1},\theta_{p^*-1}}\phi^{p^*-1})(\xi,x)$ is well defined and it is a solution of (Eq-(p^*-2)) by Proposition 4.5, which is holomorphic on $S'^*_{p^*-2}\times V_{p^*-2}$ and in $\mathrm{Exp}(\kappa_{p^*-2},S'^*_{p^*-2}\times V_{p^*-2})$ by Proposition 4.4. Thus, we can inductively define $\phi^{i-1}(\xi,x)=(\mathscr{A}_{\gamma_{i-1},\gamma_i,\theta_i}\phi^i)(\xi,x)$ and show that $\phi^{i-1}(\xi, x) \in \mathrm{Exp}(\kappa_{i-1}, S'^{*}_{i-1} \times V_{i-1})$ and it is a solution of (Eq-$(i-1)$). Finally, $\phi^1(\xi, x) = (\mathscr{A}_{\gamma_1,\gamma_2,\theta_2}\phi^2)(\xi, x) \in \mathrm{Exp}(\kappa_1, S'^{*}_1 \times V_1)$ and $u(t, x) = (\mathscr{L}_{\gamma_1,\theta_1}\phi^1)(t, x)$ is the γ-sum of $\hat{u}(t, x)$ in multisector $\mathbf{S}'$ for $x \in U' := V_1$.

5. PROOF OF PROPOSITION 4.4

The aim of this section is to show Proposition 4.4. Let us introduce differential convolution operators $\tilde{\mathfrak{P}}_i(\xi, x, \partial_x)$ $(1 \leqslant i \leqslant p^* - 1)$, which we derive from $\mathfrak{P}_i(\xi, x, \partial_x)$. Define for $\tilde{\xi}$

$$\tilde{\mathfrak{P}}_i(\xi, x, \partial_x) := \sum_{(h,0')\in\mathscr{I}} \sum_{s=0}^{h} a^i_{h,s}(\xi, x) \underset{\gamma_i}{*} ((\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s \cdot) + \sum_{\alpha} \sum_{s=0}^{\alpha_0} b^i_{\alpha,s}(\xi, x) \underset{\gamma_i}{*} ((\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s \partial_x^{\alpha'} \cdot). \tag{5.1}$$

If $\tilde{\xi} = 0$, $\tilde{\mathfrak{P}}_i(\xi, x, \partial_x) = \mathfrak{P}_i(\xi, x, \partial_x)$. Consider

$$\tilde{\mathfrak{P}}_i(\xi, x, \partial_x)\phi(\xi, x) = \psi(\xi, x), \tag{$\tilde{\mathrm{E}}$q-i}$$

where $\psi(\xi, x)$ satisfies $|\psi(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma_i}$ $(\varepsilon > 0)$ on $\{\xi \in S^*; 0 < |\xi| < \rho_0\} \times U$. Concerning the existence and uniqueness of solutions of ($\tilde{\mathrm{E}}$q-i), we have

THEOREM 5.1. *Suppose that Conditions* 1 *and* 2 *hold. Let* $i \in \{1, 2, \ldots, p^* - 1\}$ *and* $S^* := S^*(\theta, \delta)$ *be a sector with* $\bar{S}^* \cap \bar{Z}_i(R) = \emptyset$ *and* $S^* \ni \tilde{\xi} \neq 0$. *Suppose* $\psi(\xi, x) \in \mathrm{Exp}(\kappa_i, S^* \times U)$. *Then the followings hold*:

(1) *There exists a solution* $\phi(\xi, x) \in \mathrm{Exp}(\kappa_i, S^* \times U')$ *of* ($\tilde{\mathrm{E}}$q-i) *with* $|\phi(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma_i}$ $(\varepsilon > 0)$ *on* $\{\xi \in S^*; 0 < |\xi| < \rho_0\} \times U'$, *where* $U' \subset U$ *is a neighborhood of* $x = 0$.

(2) *Let* $\rho > 0$ and $\phi_j(\xi, x) \in \mathscr{O}(\{\xi \in S^*; 0 < |\xi| < \rho\} \times U')$ $(j = 1, 2)$ *be solutions of* ($\tilde{\mathrm{E}}$q-i) *with* $|\phi_j(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma_i}$ $(\varepsilon > 0)$ *on* $\{\xi \in S^*; 0 < |\xi| < \rho_0\} \times U'$. *Then* $\phi_1(\xi, x) = \phi_2(\xi, x)$.

The first assertion of Theorem 5.1 is the global existence in ξ of solutions of ($\tilde{\mathrm{E}}$q-i) and the second is the local uniqueness in ξ. The proof of Theorem 5.1 is given in Section 6. Before the proof of Proposition 4.4 we give a formula. Let $c(\xi, x) \in \mathscr{O}(S^* \times U)$ with $|c(\xi, x)| \leqslant C|\xi|^{\varepsilon-\gamma}$ $(\varepsilon > 0)$ in

$\{\xi \in S^*; 0<|\xi|<\rho_0\} \times U$. Then

$$
\begin{aligned}
(c \underset{\gamma}{*} \xi^{\gamma s}\phi)(\xi, x) &= \int_0^{\xi} c((\xi^\gamma - \eta^\gamma)^{1/\gamma}, x)\eta^{\gamma s}\phi(\eta, x)\, d\eta^\gamma \\
&= \int_0^{\tilde{\xi}} c((\xi^\gamma - \eta^\gamma)^{1/\gamma}, x)\eta^{\gamma s}\phi(\eta, x)\, d\eta^\gamma \\
&\quad + \int_{\tilde{\xi}}^{\xi} c((\xi^\gamma - \eta^\gamma)^{1/\gamma}, x)\eta^{\gamma s}\phi(\eta, x)\, d\eta^\gamma
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\tilde{\xi}}^{\xi} c((\xi^\gamma - \eta^\gamma)^{1/\gamma}, x)\eta^{\gamma s}\phi(\eta, x)\, d\eta^\gamma \\
&= \int_0^{(\xi^\gamma - \tilde{\xi}^\gamma)^{1/\gamma}} c((\xi^\gamma - \eta^\gamma - \tilde{\xi}^\gamma)^{1/\gamma}, x)(\eta^\gamma + \tilde{\xi}^\gamma)^s \phi((\eta^\gamma + \tilde{\xi}^\gamma)^{1/\gamma}, x)\, d\eta^\gamma \\
&= \int_0^{\tau} c((\tau^\gamma - \eta^\gamma)^{1/\gamma}, x)(\eta^\gamma + \tilde{\xi}^\gamma)^s \phi((\eta^\gamma + \tilde{\xi}^\gamma)^{1/\gamma}, x)\, d\eta^\gamma \big|_{\tau = (\xi^\gamma - \tilde{\xi}^\gamma)^{1/\gamma}} \\
&= c(\tau, x) \underset{\gamma}{*} ((\tau^\gamma + \tilde{\xi}^\gamma)^s \phi((\tau^\gamma + \tilde{\xi}^\gamma)^{1/\gamma}, x))\big|_{\tau = (\xi^\gamma - \tilde{\xi}^\gamma)^{1/\gamma}}.
\end{aligned}
$$

Now let us show Proposition 4.4 with the aide of Theorem 5.1.

Proof of Proposition 4.4. Let $\phi^i(\xi, x) \in \mathcal{O}(S^*_{\{0\}} \times V)$, $S^*_{\{0\}} := S^* \cap \{0<|\xi|<\rho_0\}$, be a solution of (Eq-$i$). Take $0 \neq \tilde{\xi} \in S^*_{\{0\}}$. By using the above formulas and remembering $a^i_{h,h}(\xi, x) = \gamma_i^h a_h(x)\delta(\xi)$,

$$
\begin{aligned}
&\mathfrak{P}_i(\xi, x, \partial_x)\phi^i(\xi, x) \\
&= \Bigg\{ \sum_{(h,0')\in\mathscr{I}, s} a^i_{h,s}(\tau, x) \underset{\gamma_i}{*} ((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s \phi^i((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^{1/\gamma_i}, x)) \\
&\quad + \sum_{\alpha, s} b^i_{\alpha, s}(\tau, x) \underset{\gamma_i}{*} ((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s \partial_x^{\alpha'} \phi^i((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^{1/\gamma_i}, x)) \Bigg\}\Bigg|_{\tau = (\xi^{\gamma_i} - \tilde{\xi}^{\gamma_i})^{1/\gamma_i}} + h^i(\xi, x),
\end{aligned}
$$

where

$$
\begin{aligned}
h^i(\xi, x) = &\sum_{(h,0')\in\mathscr{I}_i} \sum_{s=0}^{h-1} \int_0^{\tilde{\xi}} a^i_{h,s}((\xi^{\gamma_i} - \eta^{\gamma_i})^{1/\gamma_i}, x)\eta^{\gamma_i s}\phi^i(\eta, x)\, d\eta^{\gamma_i} \\
&+ \sum_{(h,0')\in\mathscr{I}\cap\mathscr{I}_i^c} \sum_{s=0}^{h} \int_0^{\tilde{\xi}} a^i_{h,s}((\xi^{\gamma_i} - \eta^{\gamma_i})^{1/\gamma_i}, x)\eta^{\gamma_i s}\phi^i(\eta, x)\, d\eta^{\gamma_i} \\
&+ \sum_{\alpha} \sum_{s=0}^{\alpha_0} \int_0^{\tilde{\xi}} b^i_{\alpha, s}((\xi^{\gamma_i} - \eta^{\gamma_i})^{1/\gamma_i}, x)\eta^{\gamma_i s}\partial^{\alpha'}\phi_i(\eta, x)\, d\eta^{\gamma_i}.
\end{aligned}
$$

Set $\tilde{\phi}^i(\tau, x) := \phi^i((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^{1/\gamma_i}, x)$. Then

$$\mathfrak{P}_i(\xi, x, \partial_x)\phi^i(\xi, x) = \tilde{\mathfrak{P}}_i(\tau, x, \partial_x)\tilde{\phi}^i(\tau, x)\big|_{\tau=(\xi^{\gamma_i}-\tilde{\xi}^{\gamma_i})^{1/\gamma_i}} + h^i(\xi, x),$$

so $\tilde{\mathfrak{P}}_i(\tau, x, \partial_x)\tilde{\phi}^i(\tau, x)|_{\tau=(\xi^{\gamma_i}-\tilde{\xi}^{\gamma_i})^{1/\gamma_i}} = \psi^i(\xi, x) - h^i(\xi, x)$. This implies that $\tilde{\phi}^i(\tau, x)$ satisfies for $(\tau, x) \in (S^* \cap \{|\tau| < (\rho_0^\gamma - |\tilde{\xi}|^\gamma)^{1/\gamma}\}) \times V$

$$\begin{cases} \tilde{\mathfrak{P}}_i(\tau, x, \partial_x)\tilde{\phi}^i(\tau, x) = \tilde{\psi}^i(\tau, x), \\ \tilde{\psi}^i(\tau, x) = \psi^i((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^{1/\gamma_i}, x) - h^i((\tau^{\gamma_i} + \tilde{\xi}^{\gamma_i})^{1/\gamma_i}, x). \end{cases} \tag{5.2}$$

By the definition of $\psi^i(\xi, x)$ (see (Eq-i)) and $h^i(\xi, x)$ for $(\tau, x) \in S^* \times V$

$$|\tilde{\psi}^i(\tau, x)| \leqslant A'|\tau|^{\varepsilon-\gamma_i} \exp(c'|\tau|^{\kappa_i}), \qquad \varepsilon > 0. \tag{5.3}$$

It follows from Theorem 5.1 that there exists a unique solution $\phi(\tau, x) \in \mathrm{Exp}(\kappa_i, S^* \times V')$ of (5.2) such that it is coincident with $\tilde{\phi}^i(\tau, x)$ in a neighborhood of $\tau = 0$ in S^*. Hence this implies that $\phi^i(\xi, x) = \phi((\xi^{\gamma_i} - \tilde{\xi}^{\gamma_i})^{1/\gamma_i}, x)$ has the holomorphic prolongation to $S^* \times V'$ and $\phi^i(\xi, x) \in \mathrm{Exp}(\kappa_i, S^* \times V')$.

6. PROOF OF THEOREM 5.1 AND ESTIMATES

We prove Theorem 5.1 in the former part of this section and prove Lemma 4.3 in the latter part. So we assume that all the conditions of Theorem 5.1 hold in the former part. In order to show Theorem 5.1 we decompose the operator $\tilde{\mathfrak{P}}_i(\xi, x, \partial_x)$. Set

$$\begin{cases} \tilde{\mathfrak{P}}_{i,0}(\xi, x) := \sum_{(h,0')\in\mathscr{I}_i} a^i_{h,h}(\xi, x) \underset{\gamma_i}{*} ((\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^h \cdot), \\ \tilde{\mathfrak{P}}_{i,1}(\xi, x, \partial_x) := \tilde{\mathfrak{P}}_i(\xi, x, \partial_x) - \tilde{\mathfrak{P}}_{i,0}(\xi, x). \end{cases} \tag{6.1}$$

We will find that $\tilde{\mathfrak{P}}_{i,0}(\xi, x)$ is the principal term of $\tilde{\mathfrak{P}}_i(\xi, x, \partial_x)$. It follows from the definition of $a^i_{h,h}(\xi, x)$ $(= \gamma_i^h a_h(x)\delta(\xi))$ that $\tilde{\mathfrak{P}}_{i,0}(\xi, x)$ is a multiplication and

$$\tilde{\mathfrak{P}}_{i,0}(\xi, x) = \sum_{(h,0')\in\mathscr{I}_i} \gamma_i^h a_h(x)(\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^h, \tag{6.2}$$

that is, $\tilde{\mathfrak{P}}_{i,0}(\xi, x) = \underline{A}_i(\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i}, x)$ (see (2.8)). Set $V = \{x \in \mathbb{C}^d; |x| < R\}$. Then it follows from $\bar{S}^* \cap \bar{Z}_i(R) = \emptyset$ that for fixed $S^* \ni \tilde{\xi} \neq 0$

$$|\tilde{\mathfrak{P}}_{i,0}(\xi, x)| \geqslant C(1 + |\xi|)^{\gamma_i m_{i-1}} \qquad \text{for } (\xi, x) \in S^* \times V, \tag{6.3}$$

where C depends on $\tilde{\xi}$. In the proof of Theorem 5.1 inequality (6.3) is essential. We represent $\tilde{\mathfrak{P}}_{i,1}(\xi, x, \partial_x)$ in detail

$$\begin{aligned}
\tilde{\mathfrak{P}}_{i,1}(\xi, x, \partial_x) = &\sum_{(h,0')\in\mathscr{I}_i} \sum_{s=0}^{h-1} a^i_{h,s}(\xi, x) \underset{\gamma_i}{*}((\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s\cdot) \\
&+ \sum_{(h,0')\in\mathscr{I}\cap\mathscr{I}_i^c} \sum_{s=0}^{h} a^i_{h,s}(\xi, x) \underset{\gamma_i}{*}((\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s\cdot) \\
&+ \sum_{\alpha} \sum_{s=0}^{\alpha_0} b^i_{\alpha,s}(\xi, x) \underset{\gamma_i}{*}((\xi^{\gamma_i} + \tilde{\xi}^{\gamma_i})^s \partial_x^{\alpha'}\cdot) \\
= &\sum_{(h,0')\in\in\mathscr{I}_i} \sum_{s=0}^{h-1} \sum_{l=0}^{s} a^i_{h,s}(\xi, x) \underset{\gamma_i}{*} \left(\binom{s}{l} \tilde{\xi}^{\gamma_i(s-l)} \xi^{sl}\cdot \right) \\
&+ \sum_{(h,0')\in\in\mathscr{I}\cap\mathscr{I}_i^c} \sum_{s=0}^{h} \sum_{l=0}^{s} a^i_{h,s}(\xi, x) \underset{\gamma_i}{*} \left(\binom{s}{l} \tilde{\xi}^{\gamma_i(s-l)} \xi^{sl}\cdot \right) \\
&+ \sum_{\alpha} \sum_{s=0}^{\alpha_0} \sum_{l=0}^{s} b^i_{\alpha,s}(\xi, x) \underset{\gamma_i}{*} \left(\binom{s}{l} \tilde{\xi}^{\gamma_i(s-l)} \xi^{sl} \partial_x^{\alpha'}\cdot \right).
\end{aligned}$$

Hence,

$$\tilde{\mathfrak{P}}_{i,1}(\xi, x, \partial_x) = \sum{}' a^i_{h,s,l}(\xi, x) \underset{\gamma_i}{*}(\xi^{\gamma_i l}\cdot) + \sum{}'' b^i_{\alpha,s,l}(\xi, x) \underset{\gamma_i}{*}(\xi^{\gamma_i l}\partial_x^{\alpha'}\cdot), \tag{6.4}$$

where $\sum' = \sum_{(h,0')\in\mathscr{I}_i} \sum_{s=0}^{h-1} \sum_{l=0}^{s} + \sum_{(h,0')\in\mathscr{I}\cap\mathscr{I}_i^c} \sum_{s=0}^{h} \sum_{l=0}^{s}$, $\sum'' = \sum_\alpha \sum_{s=0}^{\alpha_0} \sum_{l=0}^{s}$, $a^i_{h,s,l}(\xi, x) = \binom{s}{l}\tilde{\xi}^{\gamma_i(s-l)} a^i_{h,s}(\xi, x)$ and $b^i_{\alpha,s,l}(\xi, x) = \binom{s}{l}\tilde{\xi}^{\gamma_i(s-l)}$ $b^i_{\alpha,s}(\xi, x)$. It holds that

$$\begin{cases} |a^i_{h,s,l}(\xi, x)| \leqslant \dfrac{A|\xi|^{e_h - e(i) + \gamma_i(m_i - s - 1)}}{\Gamma\left(\frac{e_h - e(i)}{\gamma_i} + m_i - s\right)}, \\[2ex] |b^i_{\alpha,s,l}(\xi, x)| \leqslant \dfrac{A|\xi|^{e_{B,\alpha} - e(i) + \gamma_i(m_i - s - 1)}}{\Gamma\left(\frac{e_{B,\alpha} - e(i)}{\gamma_i} + m_i - s\right)} \exp(c|\xi|^{\kappa_i}), \end{cases} \tag{6.5}$$

where $e_h - e(i) + \gamma_i(m_i - s) > 0$ and $e_{B,\alpha} - e(i) + \gamma_i(m_i - s) > 0$. Consequently, we have

$$\begin{aligned}
\tilde{\mathfrak{P}}_i(\xi, x, \partial_x) = \tilde{\mathfrak{P}}_{i,0}(\xi, x) &+ \sum{}' a^i_{h,s,l}(\xi, x) \underset{\gamma_i}{*}(\xi^{\gamma_i l}\cdot) \\
&+ \sum{}'' b^i_{\alpha,s,l}(\xi, x) \underset{\gamma_i}{*}(\xi^{\gamma_i l}\partial_x^{\alpha'}\cdot).
\end{aligned} \tag{6.6}$$

Next, let us define positive integers $n_{h,s,l}$ and $n_{\alpha,s,l}$ to define the iteration procedure (see (6.8)). Since γ_i is rational, so $\gamma_i = r_i/q_i$, where q_i, r_i are positive integers with $(q_i, r_i) = 1$. Put

$$\begin{cases} n_{\alpha,s,l} = \max\{q_i(e_{B,\alpha} - e(i-1) + \gamma_i(l-s)), 1\}, \\ n_{h,s,l} = \max\{q_i(e_h - e(i-1) + \gamma_i(l-s)), 1\}, \end{cases} \tag{6.7}$$

where $0 \leqslant l \leqslant s$, and $0 \leqslant s \leqslant \alpha_0$ for $n_{\alpha,s,l}$, $0 \leqslant s \leqslant h$ for $n_{h,s,l}$ with $(h, 0') \in \mathscr{I} \cap \mathscr{I}_i^c$ and $0 \leqslant s \leqslant h-1$ for $n_{h,s,l}$ with $(h, 0') \in \mathscr{I}_i$ (see (6.4)). We show the existence of a solution of ($\tilde{\mathrm{E}}$q-i) in Section 5 by iteration. We give the iteration procedure. Define $\{\varphi_n(\xi, x)\}_{n \geqslant 0}$ by the following:

$$\begin{cases} \tilde{\mathfrak{P}}_{i,0}(\xi, x)\varphi_0(\xi, x) = \psi(\xi, x), \\ \tilde{\mathfrak{P}}_{i,0}(\xi, x)\varphi_n(\xi, x) + \sum' a^i_{h,s,l}(\xi, x) \underset{\gamma_i}{*} (\xi^{\gamma_i l}\varphi_{n-n_{h,s,l}}(\xi, x)) \\ \quad + \sum'' b^i_{\alpha,s,l}(\xi, x) \underset{\gamma_i}{*} (\xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi, x)) = 0. \end{cases} \tag{6.8}$$

We have $\varphi_0(\xi, x) = \psi(\xi, x)/\tilde{\mathfrak{P}}_{i,0}(\xi, x)$ and $\{\varphi_n(\xi, x)\}_{n \geqslant 1}$ are successively determined by the second relation in (6.8). Next step is to show the convergence of $\sum_{n=0}^{\infty} \varphi_n(\xi, x)$. In order to do so we need inequalities concerning the integers $n_{\alpha,s,l}$ and $n_{h,s,l}$ defined by (6.7).

LEMMA 6.1. (1) *Suppose* $q_i(e_{B,\alpha} - e(i-1) + \gamma_i(l-s)) \geqslant 1$. *Then*

$$\begin{cases} n - n_{\alpha,s,l} + q_i(e_{B,\alpha} - e(i) + \gamma_i(m_i - s + l)) = n + q_i\gamma_i m_{i-1}, \\ -n_{\alpha,s,l}/q_i\gamma_{i-1} + |\alpha'| + l \leqslant m_{i-1} \end{cases} \tag{6.9}$$

(2) *Suppose* $r_{\alpha,s,l} := -q_i(e_{B,\alpha} - e(i-1) + \gamma_i(l-s)) \geqslant 0$. *Then*

$$n - n_{\alpha,s,l} + q_i(e_{B,\alpha} - e(i) + \gamma_i(m_i - s + l)) = n - 1 - r_{\alpha,s,l} + q_i\gamma_i m_{i-1} \tag{6.10}$$

and

$$\frac{1 + r_{\alpha,s,l}}{q_i\gamma_i} + |\alpha'| + l \leqslant m_{i-1}, \tag{6.11}$$

in particular $0 \leqslant \gamma_i m_{i-1} - (1 + r_{\alpha,s,l})/q_i < \gamma_i m_{i-1}$.

Proof. We use $e(i-1) - e(i) = \gamma_i(m_{i-1} - m_i)$ in the proof.

(1) We have $n - n_{\alpha,s,l} + q_i(e_{B,\alpha} - e(i) + \gamma_i(m_i - s + l)) = n + q_i(e(i-1) - e(i) + \gamma_i m_i) = n + q_i\gamma_i m_{i-1}$. Since $n_{\alpha,s,l}/q_i = e_{B,\alpha} - e(i-1) + \gamma_i(l-s) \geqslant$

$\gamma_{i-1}(|\alpha| - m_{i-1}) + \gamma_i(l - s)$, $0 \leqslant s \leqslant \alpha_0$ and $0 \leqslant l \leqslant s$, we have

$$\begin{aligned}-\frac{n_{\alpha,s,l}}{q_i\gamma_{i-1}} + |\alpha'| + l &\leqslant m_{i-1} - |\alpha| - \frac{\gamma_i}{\gamma_{i-1}}(l-s) + |\alpha'| + l \\ &= m_{i-1} - \frac{\gamma_i}{\gamma_{i-1}}(l-s) + l - \alpha_0 \\ &\leqslant m_{i-1} + \left(1 - \frac{\gamma_i}{\gamma_{i-1}}\right)(l-s) \leqslant m_{i-1}.\end{aligned}$$

(2) In this case $n_{\alpha,s,l} = 1$ and (6.10) easily follows. Since $e_{B,\alpha} - e(i-1) > \gamma_i(|\alpha| - m_{i-1})$ and $0 \leqslant s \leqslant \alpha_0$,

$$\frac{r_{\alpha,s,l}}{q_i\gamma_i} + |\alpha'| + l = \frac{-e_{B,\alpha} + e(i-1)}{\gamma_i} + |\alpha'| + s < m_{i-1}.$$

$m_{i-1}, r_{\alpha,s,l}, |\alpha'|, l$ and $q_i\gamma_i = r_i$ are integers, so $m_{i-1} - \left(\frac{r_{\alpha,s,l}}{q_i\gamma_i} + |\alpha'| + l\right) > 0$ means $m_{i-1} - \left(\frac{r_{\alpha,s,l}}{q_i\gamma_i} + |\alpha'| + l\right) \geqslant 1/r_i$ and (6.11) holds. From (6.11) and $r_{\alpha,s,l} \geqslant 0$, the second inequality follows. ∎

Similar inequalities hold for $n_{h,s,l}$.

LEMMA 6.2. (1) *Suppose* $q_i(e_h - e(i-1) + \gamma_i(l-s)) \geqslant 1$. *Then*

$$\begin{cases} n - n_{h,s,l} + q_i(e_h - e(i) + \gamma_i(m_i - s + l)) = n + q_i\gamma_i m_{i-1}, \\ -n_{h,s,l}/q_i\gamma_{i-1} + l \leqslant m_{i-1}. \end{cases} \tag{6.12}$$

(2) *Suppose* $r_{h,s,l} := -q_i(e_h - e(i-1) + \gamma_i(l-s)) \geqslant 0$. *Then*

$$n - n_{h,s,l} + q_i(e_h - e(i) + \gamma_i(m_i - s + l)) = n - 1 - r_{h,s,l} + q_i\gamma_i m_{i-1}, \tag{6.13}$$

$0 \leqslant (1 + r_{h,s,l})/q_i\gamma_i + l \leqslant m_{i-1}$ *and* $0 \leqslant \gamma_i m_{i-1} - (1 + r_{h,s,l})/q_i < \gamma_i m_{i-1}$.

Proof. The proofs of (1) and (6.13) are same as Lemma 6.1. Let us proceed to show the last inequalities in (2). If $e_h - e(i-1) > \gamma_i(h - m_{i-1})$, the proof of the inequalities is same as Lemma 6.1. So assume $e_h - e(i-1) = \gamma_i(h - m_{i-1})$. In this case we have $(h, 0') \in \mathscr{I}_i$ and $s \leqslant h - 1$. Hence

$$\frac{r_{h,s,l}}{q_i\gamma_i} + l = \frac{-e_h + e(i-1)}{\gamma_i} + s = -h + m_{i-1} + s < m_{i-1},$$

from which we obtain the inequalities easily. ∎

We estimate $\varphi_n(\xi, x)$ defined by (6.8) by the method of majorant functions. For formal power series of n variables $w = (w_1, w_2, \ldots, w_n)$,

$A(w) = \sum_\alpha A_\alpha w^\alpha$ and $B(w) = \sum_\alpha B_\alpha w^\alpha$, $A(w) \ll B(w)$ means $|A_\alpha| \leqslant B_\alpha$ for all $\alpha \in \mathbb{N}^n$. $A(w) \gg 0$ means $A_\alpha \geqslant 0$ for all $\alpha \in \mathbb{N}^n$.

Let us introduce a series of majorant functions $\{\Phi^{(s)}(X)\}_{s\geqslant 0}$ of one variable X,

$$\Phi^{(s)}(X) := \frac{\Gamma(s+1)}{(r-X)^{s+1}} \qquad (r>0). \tag{6.14}$$

Obviously, $\Phi^{(s)}(X) \gg 0$ and $\frac{d\Phi^{(s)}(X)}{dX} = \Phi^{(s+1)}(X)$. We have

LEMMA 6.3. (1) *The following inequalities hold*:

$$\begin{cases} \dfrac{\Phi^{(s)}(X)}{\Gamma(s+1)} \ll \dfrac{\Phi^{(s')}(X)}{\Gamma(s'+1)} & \text{for } 0\leqslant s\leqslant s' \text{ and } 0<r\leqslant 1, \\ (R-X)^{-1}\Phi^{(s)}(X) \ll \dfrac{1}{R-r}\,\Phi^{(s)}(X) & \text{for } r<R. \end{cases} \tag{6.15}$$

In the following put $X = \sum_{i=1}^d x_i$ *and* $0<r<R$.

(2) *Let* $a(x)$ *be a holomorphic function in* $\{x; |x|<R\}$ *with* $|a(x)| \leqslant M$ *and* $v(x) \ll K\Phi^{(s)}(X)$. *Then* $a(x) \ll MR/(R-X)$ *and there is a constant* C *such that*

$$a(x)\partial_x^{\alpha'} v(x) \ll KC\Psi^{(s+|\alpha'|)}(X). \tag{6.16}$$

(3) *Let* $0<r<R$, $s_1, s_2>0$ *and* $0<\gamma\leqslant\kappa$. *Suppose that for* ξ *with* $\arg\xi = \theta$

$$\begin{cases} a(\xi,x) \ll \dfrac{Ae^{c|\xi|^\kappa}|\xi|^{s_1-\gamma}}{\Gamma(s_1/\gamma)}\,(R-X)^{-1}, \\ \varphi(\xi,x) \ll \dfrac{Ke^{c|\xi|^\kappa}|\xi|^{s_2-\gamma}}{\Gamma(s_2/\gamma)}\,\Phi^{(s)}(X). \end{cases} \tag{6.17}$$

Then for ξ *with* $\arg\xi = \theta$

$$a(\xi,x) \underset{\gamma}{*} \varphi(\xi,x) \ll \frac{AKe^{c|\xi|^\kappa}|\xi|^{s_1+s_2-\gamma}}{(R-r)\Gamma((s_1+s_2)/\gamma)}\Phi^{(s)}(X). \tag{6.18}$$

Proof. (1) By $1/(r-X)^{s'-s} \gg 1$ for $0<r\leqslant 1$ we have the first estimate. By $\frac{1}{(R-X)(r-X)} + \frac{1}{(R-r)(R-X)} = \frac{1}{(R-r)(r-X)}$, we have $\frac{1}{(R-X)(r-X)} \ll \frac{1}{(R-r)(r-X)}$.

(2) By Cauchy's inequality for holomorphic functions, $|\partial_x^\alpha a(0)| \leqslant M\alpha! R^{-|\alpha|}$. So $a(x) \ll M\sum_\alpha x^\alpha R^{-|\alpha|} = M\prod_{i=1}^d \left(1-\frac{x_i}{R}\right)^{-1} \ll M(1-(\sum_i^d x_i)/R)^{-1} =$

$MR(R-X)^{-1}$. Hence by (6.15)

$$a(x)\partial_x^{\alpha'} v(x) \ll MR(R-X)^{-1}\partial_x^{\alpha'} K\Psi^{(s)}(X) \ll KC\Psi^{(s+|\alpha'|)}(X),$$

where $C = MR(R-r)^{-1}$.

(3) By the same way as Lemma 1.4,

$$\begin{aligned}
&a(\xi, x) \underset{\gamma}{*} \varphi(\xi, x) \\
&\quad = \int_0^{|\xi|e^{i\theta}} a((\xi^\gamma - \eta^\gamma)^{1/\gamma}, x)\varphi(\eta, x)\, d\eta^\gamma \\
&\quad = \int_0^{|\xi|} a((|\xi|^\gamma - r^\gamma)^{1/\gamma} e^{i\theta}, x)\varphi(re^{i\theta}, x)e^{i\gamma\theta}\, dr^\gamma \\
&\quad \ll \frac{AK}{\Gamma(s_1/\gamma)\Gamma(s_2/\gamma)}\left(\int_0^{|\xi|}(|\xi|^\gamma - r^\gamma)^{s_1/\gamma-1} r^{s_2-\gamma} e^{c(|\xi|^\gamma - r^\gamma)^{\kappa/\gamma} + cr^\kappa}\, dr^\gamma\right)\frac{\Phi^{(s)}(X)}{R-X} \\
&\quad \ll \frac{AKe^{c|\xi|^\kappa}|\xi|^{s_1+s_2-\gamma}}{(R-r)\Gamma((s_1+s_2)/\gamma)}\Phi^{(s)}(X). \quad \blacksquare
\end{aligned}$$

Now we have majorant estimates of $\{\varphi_n(\xi, x)\}_{n=0,1,2,\dots}$ defined by (6.8). We choose r, R with $0<r<R\leqslant 1$ and set $X = \sum_{i=1}^d x_i$ in the following.

PROPOSITION 6.4. *There are positive constants A and B such that*

$$\varphi_n(\xi, x) \ll \frac{AB^n e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma(nm_{i-1}+1)}\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X) \qquad \textit{for } \xi \in S^*. \tag{6.19}$$

Proof. We show (6.19) by induction. The constant C means various constant and $V = \{|x|<R\} \Subset U$. It follows from $\tilde{\mathfrak{B}}_{i,0}(\xi, x)^{-1} \ll M(1+|\xi|)^{-\gamma_i m_{i-1}}(R-X)^{-1}$ by (6.3) and $\psi(\xi, x) \ll C|\xi|^{\varepsilon-\gamma_i}e^{c|\xi|^{\kappa_i}}(R-X)^{-1}$ that $\varphi_0(\xi, x) = \psi(\xi, x)/\tilde{\mathfrak{B}}_{i,0}(\xi, x) \ll A|\xi|^{\varepsilon-\gamma_i}e^{c|\xi|^{\kappa_i}}\Phi(X)/\Gamma(\varepsilon/\gamma_i)$. Let $n\geqslant 1$. Then we have by inductive hypothesis

$$\begin{aligned}
&\xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi, x) \\
&\qquad \ll \frac{AB^{n-n_{\alpha,s,l}}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+(n-n_{\alpha,s,l})/q_i+\gamma_i l-\gamma_i}}{\Gamma\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma((n-n_{\alpha,s,l})m_{i-1}+1)}\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}+(n-n_{\alpha,s,l})m_{i-1}+|\alpha'|\right)}(X).
\end{aligned}$$

Hence by (6.5) and Lemma 6.3(3)

$$
\begin{aligned}
& b^{i}_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x) \\
& \quad \ll CAB^{n-n_{\alpha,s,l}} e^{c|\xi|^{\kappa_i}} |\xi|^{\varepsilon+(n-n_{\alpha,s,l})/q_i+e_{B,\alpha}-e(i)+\gamma_i(m_i-s+l)-\gamma_i} \\
& \qquad \times \frac{\Gamma\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_i}+l+\frac{\varepsilon}{\gamma_i}\right)}{\Gamma\left(\frac{n-n_{\alpha,s,l}+q_i(e_{B,\alpha}-e(i)+\gamma_i(m_i-s+l))}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)} \\
& \qquad \times \frac{\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}+(n-n_{\alpha,s,l})m_{i-1}+|\alpha'|\right)}(X)}{\Gamma((n-n_{\alpha,s,l})m_{i-1}+1)}.
\end{aligned}
$$

By Lemma 6.3(1)

$$
\begin{aligned}
& \frac{\Gamma\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_i}+l+\frac{\varepsilon}{\gamma_i}\right)}{\Gamma\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)} \frac{\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}+(n-n_{\alpha,s,l})m_{i-1}+|\alpha'|\right)}(X)}{\Gamma((n-n_{\alpha,s,l})m_{i-1}+1)} \\
& \quad \ll C\frac{\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}+(n-n_{\alpha,s,l})m_{i-1}+|\alpha'|+l\right)}(X)}{\Gamma((n-n_{\alpha,s,l})m_{i-1}+1)} \\
& \quad \ll C\frac{\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}+(n-1)m_{i-1}+|\alpha'|+l\right)}(X)}{\Gamma((n-1)m_{i-1}+1)}
\end{aligned}
$$

and we have

$$
\begin{aligned}
& b^{i}_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x) \\
& \quad \ll CAB^{n-n_{\alpha,s,l}} e^{c|\xi|^{\kappa_i}} |\xi|^{\varepsilon+(n-n_{\alpha,s,l})/q_i+e_{B,\alpha}-e(i)+\gamma_i(m_i-s+l)-\gamma_i} \\
& \qquad \times \frac{\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}+(n-1)m_{i-1}+|\alpha'|+l\right)}(X)}{\Gamma\left(\frac{n-n_{\alpha,s,l}+q_i(e_{B,\alpha}-e(i)+\gamma_i(m_i-s+l))}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma((n-1)m_{i-1}+1)}. \qquad (6.20)
\end{aligned}
$$

Suppose $q_i(e_{B,\alpha} - e(i-1) + \gamma_i(l-s)))\geqslant 1$. Then by Lemma 6.1

$$
\begin{aligned}
&b^i_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x)\\
&\quad\ll \frac{CAB^{n-n_{\alpha,s,l}}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i+\gamma_i m_{i-1}-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)} \frac{\Phi^{\left(\frac{n-n_{\alpha,s,l}}{q_i\gamma_{i-1}}++(n-1)m_i+|\alpha'|+l\right)}(X)}{\Gamma((n-1)m_{i-1}+1)}\\
&\quad\ll \frac{CAB^{n-n_{\alpha,s,l}}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i+\gamma_i m_{i-1}-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)} \frac{\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X)}{\Gamma((n-1)m_{i-1}+1)}\\
&\quad\ll \frac{CAB^{n-n_{\alpha,s,l}}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i+\gamma_i m_{i-1}-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)} \frac{\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X)}{\Gamma(nm_{i-1}+1)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\tilde{\mathfrak{P}}_{i,0}(\xi,x)^{-1}(b^i_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x))\\
&\quad\ll \frac{CAB^{n-1}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)} \frac{\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X)}{\Gamma(nm_{i-1}+1)}.
\end{aligned}
\tag{6.21}
$$

Suppose $q_i(e_{B,\alpha} - e(i-1) + \gamma_i(l-s)))\leqslant 0$. Then $n_{\alpha,s,l} = 1$. By (6.20) and Lemma 6.1(2)

$$
\begin{aligned}
&b^i_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x)\\
&\quad\ll \frac{CAB^{n-1}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+(n-1-r_{\alpha,s,l})/q_i+\gamma_i m_{i-1}-\gamma_i}}{\Gamma\left(\frac{n-1-r_{\alpha,s,l}}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)\Gamma((n-1)m_{i-1}+1)} \Phi^{\left(\frac{n-1}{q_i\gamma_{i-1}}+(n-1)m_{i-1}+|\alpha'|+l\right)}(X)\\
&\quad\ll \frac{CAB^{n-1}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+(n-1-r_{\alpha,s,l})/q_i+\gamma_i m_{i-1}-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)\Gamma((n-1)m_{i-1}+1)} \Phi^{\left(\frac{n-1}{q_i\gamma_{i-1}}+(n-1)m_{i-1}+\frac{1+r_{\alpha,s,l}}{q_i\gamma_i}+|\alpha'|+l\right)}(X)\\
&\quad\ll \frac{CAB^{n-1}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+(n-1-r_{\alpha,s,l})/q_i+\gamma_i m_{i-1}-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)\Gamma((n-1)m_{i-1}+1)} \Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X),
\end{aligned}
$$

where we use $\dfrac{\Phi^{\left(\frac{n-1}{q_i\gamma_{i-1}}+(n-1)m_{i-1}+|\alpha'|+l\right)}(X)}{\Gamma\left(\frac{n-1-r_{\alpha,s,l}}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)} \ll C\,\dfrac{\Phi^{\left(\frac{n-1}{q_i\gamma_{i-1}}+(n-1)m_{i-1}+\frac{1+r_{\alpha,s,l}}{q_i\gamma_i}+|\alpha'|+l\right)}(X)}{\Gamma\left(\frac{n}{q_i\gamma_i}+m_{i-1}+\frac{\varepsilon}{\gamma_i}\right)}$. It holds by Lemma 6.1(2) that $|\xi|^{\gamma_i m_{i-1}-(1+r_{\alpha,s,l})/q_i}/(1+|\xi|)^{\gamma_i m_{i-1}} \leqslant C$, so

$$\begin{aligned}
&\tilde{\mathfrak{P}}_{i,0}(\xi,x)^{-1}(b^i_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x)) \\
&\quad\ll \frac{CAB^{n-1}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma(nm_{i-1}+1)}\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X).
\end{aligned}\tag{6.22}$$

By the same way we have from Lemma 6.2

$$\begin{aligned}
&\tilde{\mathfrak{P}}_{i,0}(\xi,x)^{-1}(a^i_{h,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{h,s,l}}(\xi,x)) \\
&\quad\ll \frac{CAB^{n-1}e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma(nm_{i-1}+1)}\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X).
\end{aligned}\tag{6.23}$$

Thus

$$\begin{aligned}
\varphi_n(\xi,x) &= -\sum{}' \tilde{\mathfrak{P}}_{i,0}(\xi,x)^{-1}(a^i_{h,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{h,s,l}}(\xi,x)) \\
&\quad - \sum{}'' \tilde{\mathfrak{P}}_{i,0}(\xi,x)^{-1}(b^i_{\alpha,s,l}(\xi,x) \underset{\gamma_i}{*} \xi^{\gamma_i l}\partial_x^{\alpha'}\varphi_{n-n_{\alpha,s,l}}(\xi,x)) \\
&\ll \frac{AB^n e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_i}\right)\Gamma(nm_{i-1}+1)}\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X). \quad \blacksquare
\end{aligned}$$

COROLLARY 6.5. *There are positive constants A, B and a polydisk U' centered at $x=0$ in $\mathbb{C}^{d+1}$ such that*

$$|\varphi_n(\xi,x)| \leqslant \frac{AB^n e^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}\Gamma\left(\frac{n}{q_i\gamma_{i-1}}+1\right)}{\Gamma\left(\frac{n}{q_i\gamma_i}+1\right)} \qquad \text{for } (\xi,x)\in S^*\times U'. \tag{6.24}$$

Proof. Let $|X|\leqslant r/2$. Then $|\Psi^{(s)}(X)|\leqslant 2^{s+1}\Gamma(s+1)/r^{s+1}$. Set $U'=\{x;|x|<r/2d\}$. Then $|X|\leqslant \sum_{i=1}^{d}|x_i|<r$ for $x\in U'$ and

$$|\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X)|\leqslant (2/r)^{\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}+1}\Gamma\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}+1\right)$$
$$\leqslant C_0C_1^n\Gamma\left(\frac{n}{q_i\gamma_{i-1}}+1\right)\Gamma(nm_{i-1}+1).$$

By the above inequality and (6.19) we have (6.24). ∎

Let us return to the proof of Theorem 5.1.

Proof of Theorem 5.1. Firstly let us show (1), the convergence and growth estimate of $\phi(\xi,x)=\sum_{n=0}^{+\infty}\varphi_n(\xi,x)$. By Corollary 6.5 and $1/\kappa_i=1/\gamma_i-1/\gamma_{i-1}$ there exists a constant $c'>0$ such that

$$|\phi(\xi,x)|\leqslant\sum_{n=0}^{+\infty}|\varphi_n(\xi,x)|\leqslant A|\xi|^{\varepsilon-\gamma_i}e^{c|\xi|^{\kappa_i}}\sum_{n=0}^{+\infty}\frac{B^n|\xi|^{n/q_i}\Gamma\left(\frac{n}{q_i\gamma_{i-1}}+1\right)}{\Gamma\left(\frac{n}{q_i\gamma_i}+1\right)}$$
$$\leqslant A|\xi|^{\varepsilon-\gamma_i}e^{c|\xi|^{\kappa_i}}\sum_{n=0}^{+\infty}\frac{B_1^n|\xi|^{n/q_i}}{\Gamma\left(\frac{n}{q_i}\left(\frac{1}{\gamma_i}-\frac{1}{\gamma_{i-1}}\right)+1\right)}\leqslant A|\xi|^{\varepsilon-\gamma_i}e^{c'|\xi|^{\kappa_i}}.$$

Secondly let us show (2), the uniqueness of solution of ($\tilde{\mathrm{E}}$q-i). Let $\phi_j(\xi,x)$ be solutions of ($\tilde{\mathrm{E}}$q-i) in $\{\xi\in S^*;0<|\xi|<\rho\}\times U'$, $j=1,2$. Set $\phi(\xi,x)=\phi_1(\xi,x)-\phi_2(\xi,x)$. Then

$$\tilde{\mathfrak{P}}_{i,0}(\xi,x)\phi(\xi,x)+\sum{}'\,a^i_{h,s,l}(\xi,x)\underset{\gamma_i}{*}(\xi^{\gamma_i l}\phi(\xi,x))$$
$$+\sum{}''\,b^i_{\alpha,s,l}(\xi,x)\underset{\gamma_i}{*}(\xi^{\gamma_i l}\partial_x^{\alpha'}\phi(\xi,x))=0. \tag{6.25}$$

We have by the same way as Proposition 6.4 for any $n\in\mathbb{N}$,

$$\phi(\xi,x)\ll\frac{AB^ne^{c|\xi|^{\kappa_i}}|\xi|^{\varepsilon+n/q_i-\gamma_i}}{\Gamma\left(\frac{n}{q_i\gamma_i}+\frac{\varepsilon}{\gamma_1}\right)\Gamma(nm_{i-1}+1)}\Phi^{\left(\frac{n}{q_i\gamma_{i-1}}+nm_{i-1}\right)}(X)\qquad\text{for }\xi\in S^*.$$

Hence, letting $n\to+\infty$, $\phi(\xi,x)\equiv 0$.

Finally, we prove Lemma 4.3. For this purpose we show the following Propositon 6.6. Let us consider the mth order formal partial differential operator $\hat{L}(t,x,\partial_t,\partial_x)$ with coefficients in $\mathcal{O}(U)[[t]]$, $U=\{x\in\mathbb{C}^d\colon |x|<R'\}$,

$0 < R' \leqslant 1$,

$$\begin{cases} \hat{L}(t, x, \partial_t, \partial_x) = L_0(t, x, \partial_t) + L_1(t, x, \partial_t, \partial_x), \\ L(t, x, \partial_t) = \sum_{h=0}^{k} c_h(x)(t\partial_t)^h, \\ \hat{L}_1(t, x, \partial_t, \partial_x) = \sum_{\alpha} t^{e_\alpha} \hat{c}_\alpha(t, x)(t\partial_t)^{\alpha_0}\partial_x^{\alpha'}, \end{cases} \tag{6.26}$$

where $m \geqslant k$ and e_α is a positive integer. Let $\hat{c}_\alpha(t, x) = \sum_{j=0}^{\infty} c_{\alpha,j}(x)t^j \in \mathcal{O}(U)[[t]]$. We assume that $\hat{L}(t, x, \partial_t, \partial_x)$ satisfies the following conditions: There is a positive constant γ such that

$$e_\alpha \geqslant \gamma(|\alpha| - k) \qquad \text{and} \qquad |c_{\alpha,j}(x)| \leqslant C_0 C^j \Gamma\left(\frac{j}{\gamma} + 1\right) \tag{6.27}$$

and $c_k(x) \neq 0$ in U.

Proposition 6.6. *Suppose* $\hat{f}(t, x) = \sum_{n=1}^{\infty} f_n(x)t^n \in t\mathcal{O}(U)[[t]]$ *satisfies* $|f_n(x)| \leqslant C^n \Gamma(n/\gamma)$. *Let* $\hat{u}(t, x) = \sum_{n=1}^{\infty} u_n(x)t^n \in t\mathcal{O}(U)[[t]]$ *be a formal solution of*

$$\hat{L}(t, x, \partial_t, \partial_x)\hat{u}(t.x) = \hat{f}(t, x) \in t\mathcal{O}(U)[[t]]. \tag{6.28}$$

Then there are constants A and B such that

$$u_n(x) \ll \frac{AB^{n-1}}{\Gamma(nk + 1)} \Phi^{(n/\gamma + nk)}(X). \tag{6.29}$$

In particular, there is a small neighborhood U' of $x = 0$ such that

$$|u_n(x)| \leqslant \frac{C^n}{\Gamma\left(\frac{n}{\gamma}\right)} \qquad \textit{for } x \in U'. \tag{6.30}$$

Proof. Set $\ell(x, n) = \sum_{h=0}^{k} c_h(x)n^h$, which is a polynomial in n with degree k and it follows from $c_k(x) \neq 0$ for $|x| < R'$ that there is an integer $n_0 \geqslant 1$ such that $|\ell(x, n)| \geqslant C_1(1 + |n|)^k$ on $\{|x| \leqslant R\}$, $R < R'$, for $n \geqslant n_0$. The coefficients $u_n(x)$ $(n \geqslant 1)$ satisfy

$$\ell(x, n)u_n(x) + \sum_{n' + j + e_\alpha = n} c_{\alpha,j}(x)n'^{\alpha_0}\partial_x^{\alpha'}u_{n'}(x) = f_n(x). \tag{6.31}$$

Let $0 < r < R < R' \leqslant 1$ and we may assume (6.29) holds for $1 \leqslant n < n_0$ and show (6.29) by induction on n. In the following C' means various constants. We note

$$\begin{aligned} \ell(x,n) &\ll C'(1+|n|)^{-k}(R-X)^{-1} \qquad \text{for } n \geqslant n_0, \\ c_{\alpha,j}(x) &\ll C_0 C^j \Gamma\left(\frac{j}{\gamma}+1\right)(R-X)^{-1}, \\ f_n(x) &\ll \frac{C^n}{\Gamma(nk+1)} \Phi^{(n/\gamma+nk)}(X). \end{aligned} \tag{6.32}$$

Since $n'/\gamma = (n-j-e_\alpha)/\gamma \leqslant (n-j)/\gamma - |\alpha| + k$,

$$\begin{aligned} c_{\alpha,j}(x) n'^{\alpha_0} \partial_x^{\alpha'} u_{n'}(x) &\ll \frac{AB^{n'-1} C_0 C^j n'^{\alpha_0}}{\Gamma(n'k+1)} \Phi^{(n'/\gamma+n'k+|\alpha'|+j/\gamma)}(X) \\ &\ll \frac{AB^{n'-1} C_0 C^j C'}{\Gamma(n'k+1)} \Phi^{(n'/\gamma+n'k+|\alpha|+j/\gamma)}(X) \\ &\ll \frac{AB^{n-e_\alpha-1}(C/B)^j C'}{\Gamma(n'k+1)} \Phi^{(n/\gamma+(n'+1)k)}(X) \\ &\ll \frac{AB^{n-e_\alpha-1}(C/B)^j C'}{\Gamma((n-1)k+1)} \Phi^{(n/\gamma+nk)}(X). \end{aligned}$$

Hence by the estimate of $\ell(x,n)$ in (6.32) and by choosing B such that $C/B \leqslant 1/2$,

$$\ell(x,n)^{-1}\left(\sum_{n'+j+e_\alpha=n} c_{\alpha,j}(x) n'^{\alpha_0} \partial_x^{\alpha'} u_{n'}(x)\right) \ll \frac{AB^{n-2}C'}{\Gamma(nk+1)} \Phi^{(n/\gamma+nk)}(X)$$

and by the estimate of $f_n(x)$ in (6.32) we have (6.29). The proof (6.30) is the same as Corollary 6.5. ∎

Proof of Lemma 4.3. Let us return to (4.4). Set

$$\begin{cases} \hat{L}(t,x,\partial_t,\partial_x) = \hat{P}(t,x,\partial_t,\partial_x), \\ L_0(t,x,\partial_t) = \sum_{h \in \mathscr{I}_{p^*-1}} a_h(x)(t\partial_t)^h, \\ \hat{L}_1(t,x,\partial_t,\partial_x) = \hat{P}(t,x,\partial_t,\partial_x) - L_0(t,x,\partial_t), \end{cases} \tag{6.33}$$

$k = m_{p^*-1}$ and $\gamma = \gamma_{p^*-1}$. Then we can apply Proposition 6.6 to the formal solutions of (EQ) and get the estimate.

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